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EXTRAPOLATION METHODS OF COMPUTING THE SADDLE POINT OF A LAGRANGE FUNCTION AND APPLICATION TO PROBLEMS WITH BLOCK-SEPARABLE STRUCTURE[1]

A.S. ANTIPIN

Consider the problem of convex block-separable programming

$$x^* \in \text{argmin} \left\{ \sum_{t=1}^{N} f_t(x_t) : \sum_{t=1}^{N} g_t(x_t) \le 0, \quad x_i \in Q_i \right\}. \tag{1}$$

Here, for any $t = 1, 2, \ldots, N$, the vector x_t belongs to an n-dimensional Euclidean space $\mathbb{R}^n$, the function $f_t(x_t)$ are convex, $g_t(x_t) = g_{1t}(x_t), \ldots, g_{mt}(x_t)$ is an m-dimensional vector, every component of which $g_{it}(x(t)$, $i = 1, 2, \ldots, m$, is a convex function in $\mathbb{R}^n$, $Q_t \subset \mathbb{R}^n$. Any solution, i.e., the vector x_t^*, of problem (1) contains $n \times N$ components and belongs to the Cartesian product of the N spaces $\mathbb{R}^n \times \ldots \times \mathbb{R}^n$.

Our aim is to decompose the initial problem (1) of high dimensionality into a series of subproblems of lower dimensionality and subsequently solve the latter independently of one another.

We propose two iterative (direct and dual) algorithms for seeking the solution of problem (1). The first iterative (dual) process is described by the following recurrence relations: let p^n, x_t^n, $t = 1, 2, \ldots, N$, be the approximation obtained; then

$$\bar{p} = \pi_+ \left(p^n + k \sum_t^N g_t(x_t^n) \right), \tag{2}$$

$$x_t^{n+1} \in \arg\min\{1/2|x_t - x_t^n|^2 + k(f_t(x_t) + (\bar{p}^n, g_t(x_t))) : x_t \in Q_t\}, \tag{3}$$

$$p^{n+1} = \pi_+ \left(p^n + k \sum_t^N g_t(x_t^{n+1}) \right). \tag{4}$$

Here, $\pi_+(a)$ is the operator of projection of a vector a onto the positive octant, i.e., $(\pi_+(a))_i = \max(0, a_i)$, $i = 1, 2, \ldots, m$. Relation (2) enables us to compute the extrapolation or predictive "price vector" $\bar{p}$, with the aid of which the iterative step to the point $x_t^{n+1}, p^{n+1})$ is realized.

The second (direct) iterative process is given by the recurrence relations

$$\bar{x}_t^n \in \arg\min\{1/2|x_t - x_t^n|^2 + k(f_t(x_t) + (p^n, g_t(x_t)) : x_t \in Q_t\}, \tag{5}$$

$$p^{n+1} = \pi_+ \left(p^n + k \sum_t^N g_t(\bar{x}_t^n) \right), \tag{6}$$

$$x_t^{n+1} \in \arg\min\{1/2|x_t - x_t^n|^2 + k(f_t(x_t) + (p^{n+1}, g_t(x_t))) : x_t \in Q_t\}. \tag{7}$$

Using subproblem (5) we compute the predictive value of direct variables x_t^n. Relations (6), (7) describe the transition to the point (x_t^{n+1}, p^{n+1}).

Theorem 1. *If problem* (1) *has a non-empty set of saddle points* $X^* \times p^* \ne \emptyset$, *while the functions* $f_t(x_t)$, $g_t(x_t) = (g_{1t}(x_t), \ldots, g_{mt}(x_t))$ *are convex, and the vector function* $g_t(x_t)$ *satisfies a Lipschitz condition with unit constant* $|g|$ *for all* $t = 1, 2, \ldots, N$ *(i.e.,* $|g_t(x_t) - g_t(y_t)| \le |g|\,|x_t - y_t|$*), the sets* Q_t *are convex and closed, and the parameter* k *is chosen from the condition* $k < (2^{1/2}|g|N)^{-1}$, *where* N *is the dimensionality of the decomposition, equal to the number of independent subproblems, then the sequence* x^n, p^n, *computed by* (2)–(4) *and* (5)–(7), *is convergent (monotonically with respect to the norm of product* N *of spaces* $\mathbb{R}^n \times \ldots \times \mathbb{R}^n$ *to the saddle point of the Lagrange function of problem* (1), *i.e.,* $x_t^n \to x_t^* \in X_t^*$, $p^n \to p^* \in P^*$, $n \to \infty$, $t = 1, 2, \ldots, N$.

The conditions of the theorem do not require the differentiability and smoothness of the functions $F_t(x_t)$ and $g_t(x_t)$. This means that processes (2)–(4) and (5)–(7) are methods of undifferentiable optimization.

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[1] Zh. vychisl. Mat. mat. Fiz., **26**, 1, pp. 150–151, 1986
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