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Solution Methods for Variational Inequalities with Coupled Constraints

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Variational inequalities with coupled constraints are considered. The class of symmetric vector functions that form coupled constraints is introduced. Explicit and implicit prediction-type gradient and proximal methods are proposed for solving variational inequalities with coupled constraints. The convergence of the methods is proved.

1. STATEMENT OF THE PROBLEM

To solve a variational inequality with coupled constraints means to find a vector $v^* \in \Omega_0$ such that

$$\langle F(v^*), w - v^* \rangle \geq 0 \qquad \forall w \in \Omega_0, \;\; g(v^*, w) \leq 0. \tag{1.1}$$

where $F(v) : R^n \to R^n$, $g(v, w) : R^n \times R^n \to R^m$, and $\Omega_0 \in R^n$ is a convex closed set.

The major difference of this setting from the conventional one lies in functional constraints of the form $g(v, w) \leq 0$, which relate the parameters of the problem and its variables. The presence of coupled constraints makes these problems more difficult to solve. However, elaborated mathematical models always involve coupled constraints. This motivates the considerable interest of specialists in such problems. There are very few publications dealing with methods for solving variational inequalities with coupled constraints. In fact, all the literature is based on the paper by Rosen [1]. In contrast, there is an extensive literature concerned with the conventional statement of the problem of a variational inequality, including solution techniques. Note, for example, a popular survey given in [2].

Problems with coupled constraints arise in many fields of mathematics. Among these are economic equilibrium models, which involve, by definition, budget constraints, according to which the scalar product of the price vector and the commodity vector does not exceed a priori costs. By their nature, these constraints are always coupled [3]. In general setting, an n-person game also leads to variational inequalities with coupled constraints [4]. Coupled constraints naturally arise in equilibrium programming problems [5] and hierarchical programming problems [6]. The development of this subject in applications of mathematical physics, where variational inequalities arose for the first time, leads to inequalities with coupled constraints [7]. This short list of problems shows that coupled constraints are characteristic of a wide class of problems rather than being an attribute of a certain one problem. For this reason, the development of methods for problems with coupled constraints is a very important task. In this paper, we propose three quite distinct methods for variational inequalities with coupled constraints and prove their convergence.

2. BASIC PROBLEMS

Below is a brief survey of most widely known problems in which coupled constrains are implied by their statement.

2.1. Two-person game with coupled constraints For simplicity, we consider a two-person game with coupled scalar constraints (see [8, 4]):

$$\begin{aligned} x_1^* &\in \operatorname{Argmin}\{f_1(x_1, x_2^*) \mid g_1(x_1, x_2^*) \le g_1(x_1^*, x_2^*), \quad x_1 \in Q_1\}, \\ x_2^* &\in \operatorname{Argmin}\{f_2(x_1^*, x_2) \mid g_2(x_1^*, x_2) \le g_2(x_1^*, x_2^*), \quad x_2 \in Q_2\}, \end{aligned} \tag{2.1}$$

where $f_1, f_2, g_1, g_2 : R^n \times R^n \to R^1$; f_1 and g_1 are convex in x_1 for any x_2; and f_2 and g_2 are convex in x_2 for any x_1.

Any n-person game can always be scalarized and reduced to the problem of calculating a fixed point of an extremal mapping. This procedure was first described in [9] for a game without coupled functional constraints. However, this procedure can be extended to games with coupled constraints. This can be done as follows. Introducing two normalized functions of the form

$$\Phi(v, w) = f_1(x_1, y_2) + f_2(y_1, x_2), \quad G(v, w) = g_1(x_1, y_2) + g_2(y_1, x_2),$$

(where $v = (y_1, y_2)$, $w = (x_1, x_2)$, $v, w \in \Omega_0 = Q_1 \times Q_2$), we state the following problem: find a vector v^* satisfying the extremal inclusion

$$v^* \in \operatorname{Argmin}\{\Phi(v^*, w) \mid G(v^*, w) - G(v^*, v^*) \le 0 \quad w \in \Omega_0\}. \tag{2.2}$$

Let us show that any solution to problem (2.2) is also a solution to (2.1).

Indeed, problem (2.2) is equivalent to the inequality

$$f_1(x_1^*, x_2^*) + f_2(x_1^*, x_2^*) \le f_1(x_1, x_2^*) + f_2(x_1^*, x_2)$$

for all x_1 and x_2 such that

$$g_1(x_1, x_2^*) + g_2(x_1^*, x_2) - g_1(x_1^*, x_2^*) - g_2(x_1^*, x_2^*) \le 0 \quad \forall x_1 \in Q_1, \ x_2 \in Q_2.$$

In particular, this system of inequalities holds for all pairs of the form $x_1, x_2^* \in Q_1 \times x_2^*$. This means that the system takes the form

$$f_1(x_1^*, x_2^*) \le f_1(x_1, x_2^*)$$

for all x_1 and x_2 satisfying

$$g_1(x_1, x_2^*) \le g_1(x_1^*, x_2^*) \quad \forall x_1 \in Q_1.$$

Since this set includes x_1^*, the last system of inequalities is obviously equivalent to the first problem in (2.1). Similar reasoning for x_1^*, x_2 leads to the second problem in (2.1).

When the objective function is differentiable, it is easy to see that problem (2.2) can always be represented in the form of a variational inequality as

$$\langle \nabla_w \Phi(v^*, v^*), w - v^* \rangle \ge 0 \quad \forall w \in \Omega_0, \quad G(v^*, w) \le G(v^*, v^*),$$

where $\nabla_w \Phi(v, v) = \nabla_w \Phi(v, w)|_{v=w}$.

2.2. The simplest model of price equilibrium. Consider the simplest market with one aggregate consumer [10]. Let $f(x)$ be his utility function, β be a fixed amount of money available to the consumer, and x be the vector of resources he wants to buy. The cost of the resources is described by the price vector p. On the one hand, the consumer cannot buy commodities whose cost is greater than β and, on the other hand, he cannot buy an amount of commodities greater than that available on the market, namely, greater than a vector y_0. Thus, assuming that the consumer is maximizing his utility function when buying commodities, we arrive at the following problem: find an equilibrium price vector $p = p^*$ and an optimal resource vector $x = x^*$ such that

$$x^* \in \text{Argmax}\{f(x) \mid \langle p^*, x\rangle \leq \beta, \quad x \in Q\}, \qquad x^* \leq y_0. \tag{2.3}$$

If the material balance $x^* \leq y_0$ in this problem is strengthened by the financial balance $\langle p^*, x^* - y_0\rangle = 0$, then these conditions satisfy an inequality of the form $\langle p - p^*, x^* - y_0\rangle \leq 0$ $\forall p \geq 0$. This means that the nonpositive linear functional $\langle p - p^*, x^* - y_0\rangle$ attains its maximum at the point p^* on the positive orthant. In other words, we have the problem

$$\begin{array}{rcl} x^* & \in & \text{Argmax}\{f(x) \mid \langle p^*, x\rangle \leq \beta, \quad x \in Q\}, \\ p^* & \in & \text{Argmax}\{\langle p - p^*, x^* - y_0\rangle \mid p \geq 0\}, \end{array}$$

whose solution is a solution to (2.3). This problem is of type (2.1).

The aggregate producer on the market under consideration is represented by the vector y_0. However, his presence on the market can be substantially strengthened if he has the possibility of minimizing, at given prices, the production of commodities that will never be bought. Thus, we obtain a model of the situation

$$\begin{array}{rcl} x^* & \in & \text{Argmax}\{f(x) \mid \langle p^*, x\rangle \leq \beta, \quad x \in Q\}, \\ y^* & \in & \text{Argmin}\{\langle p^*, y\rangle \mid x^* \leq y, \quad y \in Y\}, \end{array} \tag{2.4}$$

where Y is the set of feasible production plans. In the general case, at arbitrary prices p, the feasible production set may be empty. Therefore, it is required to choose prices $p = p^*$ such that the set

$$\{y \mid x^* \leq y, \quad y \in Y\} \neq \emptyset,$$

is nonempty and, hence, the problem has a solution.

2.3. A multicriteria decision making model on the subset of effective points. The specific character of multicriteria decision making [11] is that there are a set of alternatives $x \in Q$ on which a vector efficiency criterion $f(x) = (f_1(x), f_2(x), \ldots, f_m(x))$ is specified. A decision maker tries to increase each of the scalar criteria on the given set of alternatives. In the convex case, the scalarization of the vector criterion $\langle \lambda, f(x)\rangle = \sum_{i=1}^{m} \lambda_i f_i(x)$, where $\lambda \geq 0$, allows one to describe the set of optimal alternatives (Pareto set) as the set of optimal solutions to the family of scalar optimization problems $x_\lambda \in \text{Argmax}\{\langle \lambda, f(x)\rangle \mid x \in Q\}$ [12]. In the general case, the task of multicriteria decision making is to choose a value of $\lambda = \lambda^*$ and the corresponding optimal solution x^* such that both vectors belong to a prescribed subset of effective points; i.e.,

$$x^* \in \text{Argmax}\{\langle \lambda^*, f(x)\rangle \mid x \in Q\}, \qquad g(x^*, \lambda^*) \leq 0. \tag{2.5}$$

Assuming that the vectors λ Ë $g(x,\lambda)$ have the same dimension and strengthening the requirements $g(x^*,\lambda^*) \le 0$ by the extra condition $\langle \lambda, g(x^*,\lambda^*)\rangle = 0$ we arrive, as in (2.3), at a problem whose solution also solves (2.5):

$$\begin{aligned} x^* &\in \text{Argmax}\{\langle \lambda^*, f(x)\rangle \mid x \in Q\},\\ \lambda^* &\in \text{Argmax}\{\langle \lambda-\lambda^*, g(x^*,\lambda^*)\rangle \mid \lambda \ge 0\}. \end{aligned}$$

This problem is of type (2.1).

When model (2.5) describes a large engineering project, the maximization of the vector criterion ensures the effectiveness of that project, and the conditions $g(x,\lambda) \le 0$ describe financial, ecological, and other constraints.

2.4. Quasi-variational inequalities. Consider a bilinear two-person game with coupled constraints specified by a convex, closed set $K \in Q_1 \times Q_2 \in R^n \times R^n$ [7]. Fixing any point $\bar{x} = (\bar{x}_1, \bar{x}_2) \in K$, we construct two sections $K_1(\bar{x}) = \{x_1 \in R^n \mid (x_1, \bar{x}_2) \in K\}$ and $K_2(\bar{x}) = \{x_2 \in R^n \mid (\bar{x}_1, x_2) \in K\}$ through that point and consider the game

$$\begin{aligned} x_1^* &\in \text{Argmin}\{\langle A_1x_1, x_2^*\rangle + \langle l_1, x_1\rangle \mid x_1 \in K_1(x^*)\},\\ x_2^* &\in \text{Argmin}\{\langle x_1^*, A_2x_2)\rangle + \langle l_2, x_2\rangle \mid x_2 \in K_2(x^*)\}, \end{aligned} \tag{2.6}$$

where $x^* = (x_1^*, x_2^*)$. By introducing the vector $l = (l_1, l_2)$ and the matrix $A^\top$ with entries $a_{11} = 0$, $a_{12} = A_1^\top$, $a_{21} = A_2^\top$, and $a_{22} = 0$, where $\top$ denotes the transpose, problem (2.6) can be represented as an equivalent variational inequality:

$$\langle A^\top x^*, x - x^*\rangle + \langle l, x - x^*\rangle \ge 0 \quad \forall x \in K(x^*), \tag{2.7}$$

where $K(x^*) = K_1(x^*) \times K_2(x^*)$.

When $A_1^\top$ and $A_2^\top$ are differential operators and $K \in Q_1 \times Q_2 \subseteq H^1 \times H^2$, where H^1 and H^2 are Hilbert spaces, problem (2.7) is called a quasi-variational inequality [7].

Note that, if $g_1(x_1,x_2) = g_2(x_1,x_2)$ in problem (2.1), it takes the form (2.6).

2.5. Two-level programming. The ordinary problem of finding a maximin can be viewed as the simplest hierarchical programming problem [5]. Indeed, consider the problem of finding an optimal strategy that maximizes the minimum function,

$$\max_{\text{x}}\{\min_{y} f(x,y) \mid g(x,y) \le 0,\ y \in Y\} = \max_{\text{x}}\min_{\text{y}}\{f(x,y) \mid g(x,y) \le 0,\ y \in Y\}.$$

Here $x \in X(y) \subset R^n$ and $y \in Y \subset R^n$. Any point of the manifold $y(x) = \text{Argmin}\{f(x,y) \mid g(x,y) \le 0,\ y \in Y\}$ can be a solution to this problem. However, if $f(x,y)$ and $g(x,y)$ are convex in y for any x and x^* is a fixed point of the extremal inclusion

$$x^* \in \text{Argmin}\{f(x^*,y) \mid g(x^*,y) \le 0,\ y \in Y\},$$

then the minimax problem can be reduced to finding a fixed point of this extremal mapping.

3. SYMMETRIC FUNCTIONS

Problems with coupled constraints have always attracted and attract the attention of investigators. We mention [1, 13], where gradient approaches to these problems were discussed. Game problems with coupled constraints were analyzed in [14]. The study [1] is considered to be one of the best and has often been cited until now. The basic premise of these studies is the assumption that the function $g(v,w)$ involved in constraints is jointly convex in v and w. This is a very severe requirement. It never holds for constraints in economic equilibrium models, since they involve budget constraints like $\langle p,x\rangle \le m$, where p denotes the prices, x — the commodities, and m — given costs. Here, $g(p,x)=\langle p,x\rangle$ is not jointly convex in its variables.

In this paper, we drop the requirement that $g(v,w)$ be jointly convex in v and w and, instead, use the symmetry of these functions about the diagonal of the square $\Omega_0\times\Omega_0$, i.e., about the manifold $v=w$.

Definition 1. *A function $g(v,w)$ from $I\!R^n\times I\!R^n$ into $I\!R^m$ is said to be symmetric on $R^n\times R^n$ if it satisfies the condition*

$$g(v,w)=g(w,v)\quad \forall w\in\Omega_0,\quad \forall v\in\Omega_0. \tag{3.1}$$

Examples of symmetric functions include primarily functions generating budget constraints in economic equilibrium models. They have the form $g(v,w)=\langle v,w\rangle$ or $g(v,w)=\langle Av,w\rangle$, where A is a symmetric matrix. Widely known in applications are the Cobb–Douglas production function and the constant-elasticity-of-substitution production function: $g(v,w)=Av^\alpha w^\beta$ and $g(v,w)=A(\alpha v^{-\omega}+\beta w^{-\omega})^{-\gamma/\omega}$, where $A>0$, $\alpha>0$, $\beta>0$, $\omega>0$ are parameters. If α and β are equal, then these functions are symmetric in the sense of (3.1). It is easy to see that $\Phi(v,w)=f(x_1,y_2)+f(y_1,x_2)$, where $v=(y_1,y_2)$ and $w=(x_1,x_2)$, is a symmetric function.

Let us analyze the characteristic properties of symmetric functions [15]. To this end, we differentiate (3.1) with respect to w to obtain

$$\nabla_w^\top g(v,w)=\nabla_v^\top g(w,v)\quad \forall w\in\Omega_0,\quad \forall v\in\Omega_0, \tag{3.2}$$

where $\nabla_w^\top g(v,w)$ and $\nabla_v^\top g(w,v)$ are $m\times n$ matrices whose rows are the vectors $\nabla_v g_i(w,v)$ and $\nabla_w g_i(v,w)$, $i=1,2,\dots,m$.

Setting $w=v$ in (3.2) yields

$$\nabla_w^\top g(v,v)=\nabla_v^\top g(v,v)\quad \forall v\in\Omega_0. \tag{3.3}$$

Thus, we can state the following.

Property 1. *For the gradients of a symmetric vector function $g(v,w)$ with respect to v and w, the matrices of their restrictions to the diagonal of the square $\Omega_0\times\Omega_0$ are equal.*

According to the definition of a differentiable function $g(v,w)$, we have (see [16])

$$g(v+h,w+k)=g(v,w)+\nabla_v^\top g(v,w)h+\nabla_w^\top g(v,w)k+\omega(v,w,h,k), \tag{3.4}$$

where $\omega(v,w,h,k)/(|h|^2+|k|^2)^{1/2}\to 0$ as $|h|^2+|k|^2\to 0$. Let $w=v$ and $h=k$. Then, in view of (3.3) and (3.4), we obtain

$$g(v+h,v+h)=g(v,v)+2\nabla_w^\top g(v,v)h+\omega(v,h), \tag{3.5}$$

where $\omega(v,h)/|h|\to 0$ as $|h|\to 0$. Being a special case of (3.4), formula (3.5) means that the restriction of the gradient $\nabla_w^\top g(v,w)$ to the diagonal of $\Omega_0\times\Omega_0$ is the gradient $\nabla^\top g(v,v)$ of $g(v,v)$, i.e.,

$$2\nabla_w^\top g(v,w)|_{v=w}=\nabla^\top g(v,v)\quad \forall v\in\Omega_0. \tag{3.6}$$

Thus, we have proved the following property (see [17]).

Property 2. *The operator $2\nabla_w g(v,w)|_{v=w}$ is potential and equal to the gradient of the restriction of the symmetric function $g(v,w)$ to the square's diagonal; i.e., $2\nabla_w^\top g(v,v) = \nabla^\top g(v,v)$.*

This key property of symmetric functions is crucial in the following analysis.

As mentioned above, if $g_1(x_1,x_2)$ and $g_2(x_1,x_2)$ in problem (2.1) are equal, then problem (2.1) reduces to (2.6). Let us verify that the normalized function $G(v,w)$ in (2.2) then satisfies the symmetry property (3.1). Indeed, $G(v,w) = g_1(x_1,y_2)+g_2(y_1,x_2)$ and $G(w,v) = g_1(y_1,x_2)+g_2(x_1,y_2)$. Since $g_1(x_1,x_2) = g_2(x_1,x_2)$, it is obvious that $G(v,w) = G(w,v)$. Thus, problem (2.1) in the case under consideration has symmetric coupled constraints.

4. SYMMETRIZATION

The coupled constraints in problem (1.1) may not be symmetric. For example, they may be antisymmetric, i.e., may satisfy the condition $g(v,w) = -g(w,v)\ \forall v,w \in \Omega_0$. We show that the coupled constraints then have no effect on the solution to problem (1.1) and, hence, can be dropped. Indeed, consider a pair of problems,

$$\langle F(v^*), w - v^* \rangle \geq 0, \quad \forall w \in \Omega_0,$$

and

$$\langle F(v^*), w - v^* \rangle \geq 0, \quad g(v^*,w) \leq 0, \qquad \forall w \in \Omega_0,$$

where $g(v,w)$ is an antisymmetric function. Such a function always vanishes on the diagonal of $\Omega_0 \times \Omega_0$, because setting $v = w$ in $g(v,v) = -g(v,v)$ yields $g(v,v) = 0$. Consider the intersection of two sets: $\Omega_0 \bigcap \{w \mid g(v^*,w) \leq 0\}$. It is always nonempty (contains the point v^*) and is a subset of Ω_0. Since v^* is a minimum point of $\langle F(v^*), w - v^* \rangle$ on Ω_0 (i.e., a solution to the first problem), it is also a minimum point of this function on any of its subsets; i.e., it is a solution to the second problem. Thus, antisymmetric coupled constraints in equilibrium problems can be dropped.

In the general case, when $g(v,w)$ is neither symmetric nor antisymmetric, the constraints in problem (1.1) can be symmetrized. Indeed, define two subclasses of symmetric and antisymmetric vector functions:

$$g(v,w) - g(w,v) = 0 \ \ \forall w \in \Omega_0, \quad \forall v \in \Omega_0, \tag{4.1}$$

$$g(v,w) + g(w,v) = 0 \ \ \forall w \in \Omega_0, \quad \forall v \in \Omega_0. \tag{4.2}$$

These conditions generalize the concepts of symmetric and antisymmetric matrices [17]. The transpose is defined as $g^\top(v,w) = g(w,v)$ (see [15]). In terms of this function, the conditions of symmetry (4.1) and antisymmetry (4.2) have the form

$$\Phi(v,w) = \Phi^\top(v,w), \quad \Phi(v,w) = -\Phi^\top(v,w).$$

By using the obvious relations $\Phi(v,w) = (\Phi^\top(v,w))^\top$ and $(\Phi_1(v,w)+\Phi_2(v,w))^\top = \Phi_1^\top(v,w)+\Phi_2^\top(v,w)$, it is easy to see that any real function $\Phi(v,w)$ can be decomposed as

$$g(v,w) = s(v,w) + k(v,w), \tag{4.3}$$

where $s(v,w)$ and $k(v,w)$ are a symmetric and an antisymmetric function, respectively. This decomposition is unique, and

$$s(v,w) = \frac{1}{2}(g(v,w) + g^\top(v,w)), \quad k(v,w) = \frac{1}{2}(g(v,w) - g^\top(v,w)). \tag{4.4}$$

Using this decomposition, we represent the functional constraints of problem (1.1) as $\{w \mid \mid g(v^*,w) = s(v^*,w) + k(v^*,w) \le 0,\ w \in \Omega_0\}$. Apparently, from the reasoning atthe beginning of this section, it follows that the antisymmetric part of the constraints can also be dropped in this case. Indeed, let v^* be a solution to the problem

$$\langle F(v^*), w - v^*\rangle \ge 0, \qquad s(v^*,w) \le 0, \quad w \in \Omega_0. \tag{4.5}$$

We introduce notations for the sets $D = \{w \mid g(v^*,w) \le 0,\ w \in \Omega_0\}$ and $K_1 = \{w \mid \mid k(v^*,w) \le 0,\ w \in \Omega_0\}$ and $K_2 = \{w \mid k(v^*,w) > 0,\ w \in \Omega_0\}$. The feasible set D of the original problem is partitioned into two parts, $D_1 = D \bigcap K_1$ and $D_2 = D \bigcap K_2$ such that $D = D_1 \cup D_2$. For all $w \in D_2$, the function $k(v^*,w)$ in $s(v^*,w) + k(v^*,w) \le 0$ $(w \in \Omega_0)$ can be dropped; then, $D_2 \subset \{w \mid s(v^*,w) \le 0,\ w \in \Omega_0\}$. On the other hand, consider the intersection $D_1 \bigcap \{w \mid s(v^*,w) \le 0,\ w \in \Omega_0\}$, which includes v^* and on which the function $\langle F(v^*), w - v^*\rangle$ reaches a minimum. It is easy to see that any point of this intersection satisfies $s(v^*,w) + k(v^*,w) \le 0$, $w \in \Omega_0$. Consequently, if a solution to problem (1.1) has an interior neighborhood, for example, when $g(v^*,v^*) < 0$ and $w \in \Omega_0$, then the solution to (4.5) is a solution to (1.1).

Thus, to find a solution to problem (1.1), we have to solve the symmetrized problem

$$\langle F(v^*), w - v^*\rangle, \quad g(v^*,w) + g^\top(v^*,w) \le 0 \qquad \forall w \in \Omega_0.$$

The idea of symmetrizing constraints makes it possible, in principle, to solve equilibrium problems with coupled constraints. Some considerations regarding the symmetrization of sets can also be found in [18].

5. REDUCTION TO A SADDLE-POINT PROBLEM

Problem (1.1) can always be viewed as the minimization of the linear function $f(w) = \langle F(v^*), w - v^*\rangle$ on the set $\Omega = \{w \in \Omega_0 \mid g(v^*,w) \le 0\}$. Define the Lagrange functions $\mathcal{L}(v^*,w,p) = \langle F(v^*), w - v^*\rangle + \langle p, g(v^*,w)\rangle$ $\forall w \in \Omega_0$, $\forall p \ge 0$, where v^* is a solution to the problem and w and p are the primal and dual variables. Since v^* is the minimum of $f(w)$ on Ω_0, the pair (v^*,p^*) (under certain regularity conditions) is a saddle point of $\mathcal{L}(v^*,w,p)$, i.e., satisfies the system of inequalities

$$\begin{gathered} \langle F(v^*), v^* - v^*\rangle + \langle p, g(v^*,v^*)\rangle \le \\ \le \langle F(v^*), v^* - v^*\rangle + \langle p^*, g(v^*,v^*)\rangle \le \\ \le \langle F(v^*), w - v^*\rangle + \langle p^*, g(v^*,w)\rangle \end{gathered} \tag{5.1}$$

for all $w \in \Omega_0$ and $p \ge 0$.

This system can be represented in a somewhat different manner as

$$\begin{aligned} v^* &\in \operatorname{Argmin}\{\langle F(v^*), w - v^*\rangle + \langle p^*, g(v^*,w)\rangle \mid w \in \Omega_0\}, \\ p^* &\in \operatorname{Argmax}\{\langle p, g(v^*,v^*)\rangle \mid p \ge 0\}. \end{aligned} \tag{5.2}$$

There are other equivalent representations of system (5.1). Assuming that $g(v,w)$ is differentiable with respect to w for any v, we rewrite system (5.2) as

$$\begin{array}{l}\langle F(v^*) + \nabla_w^\top g(v^*, v^*)p^*, w - v^*\rangle \geq 0 \quad \forall w \in \Omega_0, \\ \langle -g(v^*, v^*), p - p^*\rangle \geq 0 \quad \forall p \geq 0.\end{array} \tag{5.3}$$

By using a projection operator, this system of variational inequalities is represented in the form of operator equations as

$$\begin{array}{rcl} v^* & = & \pi_{\Omega_0}(v^* - \alpha(F(v^*) + \nabla_w^\top g(v^*, v^*)p^*), \\ p^* & = & \pi_+(p^* + \alpha g(v^*, v^*)), \end{array} \tag{5.4}$$

where $\pi_+(\ldots)$ and $\pi_{\Omega_0}(\ldots)$ are the operators projecting a vector onto the positive orthant R^n_+ and the set Ω_0, respectively, and $\alpha > 0$ is the step-length parameter.

System (5.3) can be transformed as follows. The first inequality in this system is represented as

$$\langle F(v^*), w - v^*\rangle + \langle p^*, \nabla_w g(v^*, v^*)(w - v^*)\rangle \geq 0 \quad \forall w \in \Omega_0.$$

Next, taking into account the key property (3.6) of symmetric functions and the convexity of $g(v,w)_{v=w}$ on the diagonal of the square $\Omega_0 \times \Omega_0$, we transform the term

$$\langle p^*, \nabla_w g(v^*, v^*)(w - v^*)\rangle = \frac{1}{2}\langle p^*, \nabla g(v^*, v^*)(w - v^*)\rangle \leq \frac{1}{2}\langle p^*, g(w, w) - g(v^*, v^*)\rangle.$$

Finally, (5.3) can be represented as

$$\begin{array}{ll} \langle F(v^*), w - v^*\rangle + \frac{1}{2}\langle p^*, g(w, w) - g(v^*, v^*)\rangle \geq 0 & \forall w \in \Omega_0, \\ \langle -g(v^*, v^*), p - p^*\rangle \geq 0 & \forall p \geq 0. \end{array} \tag{5.5}$$

Thus, the variational inequality with coupled constraints reduces to the saddle-point problem (5.5). This problem can be solved by the methods described in [19]. However, the development of methods in terms of the original problem is of considerable interest, at least, for two reasons. First, these methods are interpreted as dynamic models of matching conflicts of factors or interests. Second, these methods will be basic for various symmetrization procedures in problems with asymmetric coupled constraints.

6. THE METHOD INVOLVING THE MODIFIED LAGRANGE FUNCTION

To solve system (5.2), we consider the simple iteration method

$$\begin{array}{rcl} v^{n+1} & \in & \mathrm{argmin}\{\frac{1}{2}|w - v^n|^2 + \alpha(\langle F(v^n), w - v^n\rangle + \langle p^n, g(v^n, w)\rangle) \mid w \in \Omega_0\}, \\ p^{n+1} & = & \pi_+(p^n + \alpha g(v^{n+1}, v^{n+1})). \end{array} \tag{6.1}$$

However, it is well known that this method does not converge to solutions even for optimization problems and much less for equilibrium ones. In optimization, this situation was overcome by invoking the modified Lagrange function. Let us verify that this idea, in conjunction with the symmetry of $g(v,w)$, is also effective in our case [20, 21]. Consider the method of the modified Lagrange function as applied to variational inequalities with coupled constraints:

$$\begin{array}{rcl} v^{n+1} & \in & \mathrm{argmin}\{\frac{1}{2}|w - v^n|^2 + \alpha\mathcal{M}(v^{n+1}, w, p^n) \mid w \in \Omega_0\}, \\ p^{n+1} & = & \pi_+(p^n + \alpha g(v^{n+1}, v^{n+1})), \quad \alpha > 0, \end{array} \tag{6.2}$$

where

$$\mathcal{M}(v,w,p) = \langle F(v), w-v\rangle + \frac{1}{2\alpha}\left|\pi_+(p+\alpha g(v,w))\right|^2 - \frac{1}{2\alpha}|p|^2$$

is defined for all $v, w \in I\!R^n \times I\!R^n$ and $p \geq 0$. Here, v^n, p^n is the current approximation and v^{n+1}, p^{n+1} is the desired solution. Relations (6.2) are equations with variables v^{n+1}, which appear on the left and right of the expression (implicit scheme).

Representing (6.2) as variational inequalities yields

$$\langle v^{n+1}-v^n+\alpha(F(v^{n+1})+\nabla_w^\top g(v^{n+1},v^{n+1})\pi_+(p^n+\alpha g(v^{n+1},v^{n+1})), w-v^{n+1}\rangle \geq 0 \ \forall w \in \Omega_0, \quad (6.3)$$

$$\langle p^{n+1}-p^n-\alpha g(v^{n+1},v^{n+1}), p-p^{n+1}\rangle \geq 0 \quad \forall p \geq 0. \quad (6.4)$$

The iteration formulas of process (6.2) are equivalent to (6.3) and (6.4).

Comparing these variational inequalities with the original problem (1.1), it is important to note that the original problem with coupled constraints is replaced by a sequence of problems, each consisting of a system of two ordinary variational inequalities (without coupled constraints), for which there are a variety of solution techniques [2].

Before proving the convergence (monotone in norm) of method (6.2) to an equilibrium solution to the problem, we make an important remark. The conditions of the theorem require that $g(v,w)$ be convex only on the diagonal of $\Omega_0 \times \Omega_0$ and do not require that it be convex in w for any v. However, method (6.2) assumes the minimization of the regularized function $\mathcal{M}(v^{n+1}, w, p^n)$ in w for any v, and this function involves $g(v,w)$. The function $\frac{1}{2}|w-v^n|^2+$ $+\mathcal{M}(v^{n+1}, w, p^n)$ can be considered convex if α is sufficiently small. According to the theorem's conditions, this parameter takes any value, including sufficiently small.

Let us show that any function whose gradient satisfies the Lipschitz condition can be made convex (up- or downward). Indeed, suppose that the gradient of $f(x)$ satisfies the Lipschitz condition; i.e., $\langle \nabla f(x+h) - \nabla f(x), h\rangle \leq L|h|^2$ on some set or

$$-L|h|^2 \leq \langle \nabla f(x+h) - \nabla f(x), h\rangle \leq L|h|^2.$$

The left inequality of this system yields

$$\langle (\nabla f(x+h) + LI(x+h)) - (\nabla f(x) + LI(x), h\rangle \geq 0.$$

This means that, for all $\alpha \geq L$, the function $f_\alpha(x) = f(x) + (1/2)\alpha|x|^2$ is convex (and even strongly convex, if $\alpha > L$) on some set of x. A similar line of reasoning for the right inequality of the system shows that $f_\alpha(x) = f(x) - (1/2)\alpha|x|^2$ is concave on the same set for all $\alpha \geq L$. In other words, the regularization of a nonconvex function makes it convex for sufficiently large values of the regularization parameter.

Theorem 1. *Let the solution set of problem* (1.1) *be nonempty, $F(v)$ be a monotone operator, $g(v,w)$ be a symmetric vector function differentiable with respect to w for any v, its restriction $g(v,w)|_{v=w}$ to the square's diagonal be a convex function, $\Omega \subseteq I\!R^n$ be a convex closed set, and $\alpha > 0$. Then, the sequence v^n constructed by method* (6.2) *converges monotonically in norm to an equilibrium solution to problem* (1.1)*; i.e., $v^n \to v^* \in \Omega^*$ as $n \to \infty$.*

Proof. Setting $w = v^*$ in (6.3) and taking into account the second equation in (6.2), we obtain

$$\langle v^{n+1}-v^n+\alpha F(v^{n+1})+\alpha\nabla_w^\top g(v^{n+1},v^{n+1})p^{n+1}, v^*-v^{n+1}\rangle \geq 0.$$

This equality can be transformed as follows:

$$\begin{aligned} &\langle v^{n+1} - v^n, v^* - v^{n+1}\rangle + \alpha\langle F(v^{n+1}), v^* - v^{n+1}\rangle + \\ &+\alpha\langle \nabla_w^\top g(v^{n+1}, v^{n+1}) p^{n+1}, v^* - v^{n+1}\rangle \geq 0. \end{aligned} \tag{6.5}$$

In view of (3.6) and since $g(v, v)$ is convex, the last term in (6.5) can be transformed so as

$$\begin{aligned} \langle p^{n+1}, \nabla_w g(v^{n+1}, v^{n+1})(v^* - v^{n+1})\rangle &= \frac{1}{2}\langle p^{n+1}, \nabla g(v^{n+1}, v^{n+1})(v^* - v^{n+1})\rangle \leq \\ &\leq \frac{1}{2}\langle p^{n+1}, g(v^*, v^*) - g(v^{n+1}, v^{n+1})\rangle, \end{aligned} \tag{6.6}$$

then, we have

$$\begin{aligned} &\langle v^{n+1} - v^n, v^* - v^{n+1}\rangle + \alpha\langle F(v^{n+1}), v^* - v^{n+1}\rangle + \\ &+\frac{\alpha}{2}\langle p^{n+1}, g(v^*, v^*) - g(v^{n+1}, v^{n+1})\rangle \geq 0. \end{aligned} \tag{6.7}$$

Setting $w = v^{n+1}$ in the first inequality in (5.5) yields

$$\langle F(v^*), v^{n+1} - v^*\rangle + \frac{1}{2}\langle p^*, g(v^{n+1}, v^{n+1}) - g(v^*, v^*)\rangle \geq 0. \tag{6.8}$$

Summing (6.7) and (6.8) gives

$$\begin{aligned} &\langle v^{n+1} - v^n, v^* - v^{n+1}\rangle + \alpha\langle F(v^{n+1}) - F(v^*), v^* - v^{n+1}\rangle + \\ &+\frac{\alpha}{2}\langle p^{n+1} - p^*, g(v^*, v^*) - g(v^{n+1}, v^{n+1})\rangle \geq 0. \end{aligned} \tag{6.9}$$

Setting $p = p^*$ in (6.4) and using $\langle p^{n+1}, g(v^*, v^*)\rangle \leq 0$ and $\langle p^*, g(v^*, v^*)\rangle = 0$, we can write

$$\frac{1}{2}\langle p^{n+1} - p^n, p^* - p^{n+1}\rangle - \frac{\alpha}{2}\langle g(v^{n+1}, v^{n+1}) - g(v^*, v^*), p^* - p^{n+1}\rangle \geq 0. \tag{6.10}$$

Since $F(v)$ is a monotone operator, the summation of (6.9) and (6.10) produces

$$\langle v^{n+1} - v^n, v^* - v^{n+1}\rangle + \frac{1}{2}\langle p^{n+1} - p^n, p^* - p^{n+1}\rangle \geq 0.$$

By using the identity

$$|x_1 - x_3|^2 = |x_1 - x_2|^2 + 2\langle x_1 - x_2, x_2 - x_3\rangle + |x_2 - x_3|^2, \tag{6.11}$$

the scalar products on the left of the last inequality can be decomposed into a sum of squares:

$$|v^{n+1} - v^*|^2 + \frac{1}{2}|p^{n+1} - p^*|^2 + |v^{n+1} - v^n|^2 + \frac{1}{2}|p^{n+1} - p^n|^2 \leq |v^n - v^*|^2 + \frac{1}{2}|p^n - p^*|^2. \tag{6.12}$$

Summing (6.12) from $n = 0$ to $n = N$ yields

$$\begin{aligned} &|v^{N+1} - v^*|^2 + \frac{1}{2}|p^{N+1} - p^*|^2 + \sum_{k=0}^{k=N} |v^{k+1} - v^k|^2 + \frac{1}{2}\sum_{k=0}^{k=N} |p^{k+1} - p^k|^2 \leq \\ \leq\ &|v^0 - v^*|^2 + \frac{1}{2}|p^0 - p^*|^2. \end{aligned}$$

This inequality implies the boundedness of the trajectory; i.e.,

$$|v^{N+1} - v^*|^2 + \frac{1}{2}|p^{N+1} - p^*|^2 \leq |v^0 - v^*|^2 + \frac{1}{2}|p^0 - p^*|^2, \tag{6.13}$$

and also the convergence of the series

$$\sum_{k=0}^{\infty} |v^{k+1} - v^k|^2 < \infty, \quad \sum_{k=0}^{\infty} |p^{k+1} - p^k|^2 < \infty$$

therefore,

$$|v^{n+1} - v^n|^2 \to 0, \quad |p^{n+1} - p^n|^2 \to 0, \quad n \to \infty. \tag{6.14}$$

Since the sequence (v^n, p^n) is bounded, there exists an element (v', p') such that $v^{n_i} \to v'$ and $p^{n_i} \to p'$ as $n_i \to \infty$; moreover,

$$|v^{n_i+1} - v^{n_i}|^2 \to 0, \quad |p^{n_i+1} - p^{n_i}|^2 \to 0.$$

Considering (6.3) and (6.4) for all $n_i \to \infty$ and passing to the limit as produces

$$\begin{aligned} &\langle F(v') + \nabla_w^\top g(v', v')p', w - v'\rangle \geq 0, \quad p' = \pi_+(p' + \alpha g(v', v')), \\ &\langle -g(v', v'), p - p'\rangle \geq 0 \quad \forall p \geq 0. \end{aligned}$$

Since these relations coincide with (5.3), we have $v' = v^* \in \Omega^*$ and $p' = p^* \geq 0$; i.e., any limit point of the sequence (v^n, p^n) is a solution to the problem. The monotonic decreasing of $|v^n - v^*| + |p^n - p^*|$ ensures that the limit point is unique; i.e., $v^n \to v^*$ and $p^n \to p^*$ as $n \to \infty$. The theorem is proved. □

The above proof is a basic scheme that can be extended to approximate solutions of regularized problems and to approximately given $\Phi(v, w)$ and $g(w)$.

7. A PREDICTION-TYPE PROXIMAL METHOD WITH RESPECT TO DUAL VARIABLES

Method (6.2) is convergent due to the use of the modified Lagrange function. In many cases, however, the modified Lagrange function disturbs the problem's decomposition structure, if any. For example, the block separable structure allows one to decompose the original problem into independent subproblems, but it is lost after using the modified Lagrange function. On the other hand, the use of the ordinary Lagrange function instead of the modified one preserves the block separable structure of the problem, because the ordinary Lagrange function is a linear convolution of the objective function with functional constraints. This means that the use of the ordinary Lagrange function (instead of the modified one) in iterative methods allows one to decompose the auxiliary optimization problem into a series of independent problems of smaller sizes at every iteration step. This circumstance is of great importance for game problems, since they are, as a rule, of large size.

Consider the analogue of (6.2) based on the ordinary Lagrange function. Let (v^n, p^n) be the current approximation. Then, the next approximations (v^{n+1}, p^{n+1}) is determined by the formulas (see [20, 21])

$$\begin{aligned} \bar{p}^n &= \pi_+(p^n + \alpha_n g(v^n, v^n)), \\ v^{n+1} &= \text{argmin}\{\frac{1}{2}|w - v^n|^2 + \alpha_n L(v^{n+1}, w, \bar{p}^n) \mid w \in \Omega\}, \\ p^{n+1} &= \pi_+(p^n + \alpha_n g(v^{n+1}, v^{n+1})), \end{aligned} \tag{7.1}$$

where

$$L(v, w, p) = \langle F(v), w - v \rangle + \langle p, g(v, w) \rangle.$$

The steplength α_n in (7.1) is either determined by the condition

$$0 < \varepsilon \leq \alpha_n < \sqrt{2}/|g|, \quad \varepsilon > 0, \tag{7.2}$$

where $|g|$ is the constant defined in (7.4), or by the condition

$$\alpha_n |g(v^{n+1}, v^{n+1}) - g(v^n, v^n)| \leq \sqrt{2(1-\varepsilon)} |v^{n+1} - v^n|. \tag{7.3}$$

To verify (7.3), we first choose an arbitrary α_0 (the same for all iteration step, e.g., $\alpha_0 = 1$), then perform calculations for the first two iteration steps by formulas (7.1) (i.e., calculate the vectors $\bar{p}^n$ and v^{n+1}), and next verify the condition. If it is fulfilled, the step length is set equal to the value found; but if not, the parameter is reduced by multiplying it a number smaller than unity, etc., until condition (7.3) is satisfied.

This method for choosing the step length may seem too laborious at first glance. Indeed, to determine α_n in the general case, one has to minimize a strongly convex function on a simple set several times. However, no Lipschitz constants or upper estimates for Lagrange multipliers are required to be known at advance in this method. Moreover, new parameter values are not necessarily determined at every iteration step. It may be sufficient to use old parameter values, sometimes correcting them.

To prove the validity of the method for determining α_n by (7.2) and (7.3), we assume that the operator $g(v, v)$ satisfies the Lipschitz condition

$$|g(v + h, v + h) - g(v, v)| \leq |g| \, |h| \tag{7.4}$$

for all $w, w + h \in \Omega$, where $|g|$ is a constant. Condition (7.4) is used to estimate the difference between two vectors, $\bar{p}^n$ and p^{n+1}. In view of (7.4), we find from (7.1) that

$$\left| \bar{p}^n - p^{n+1} \right| \leq \alpha_n \left| g(v^n, v^n) - g(v^{n+1}, v^{n+1}) \right|. \tag{7.5}$$

According to (7.4), any pair of points v^{n+1} and v^n in (7.1) is subject to the condition

$$\left| g(v^{n+1}, v^{n+1}) - g(v^n, v^n) \right| \leq |g| \left| v^{n+1} - v^n \right|.$$

This inequality is always true if the parameter α_n in (7.1) is determined by the condition $\sqrt{2(1-\varepsilon)}/\alpha_n \geq |g|$ or, equivalently, by $\alpha_n \leq \sqrt{2(1-\varepsilon)}/|g|$; i.e.,

$$|g(v^{n+1}, v^{n+1}) - g(v^n, v^n)| \leq \frac{\sqrt{2(1-\varepsilon)}}{\alpha_n} |v^{n+1} - v^n|.$$

This means that there always exists α_n satisfying condition (7.3). In fact, this condition provides a way of estimating the unknown Lipschitz constant $|g|$ calculations by this method.

Method (7.1) can be represented as the variational inequalities

$$\langle v^{n+1} - v^n + \alpha_n (F(v^{n+1}) + \nabla_w^\top g(v^{n+1}, v^{n+1}) \bar{p}^n), w - v^{n+1} \rangle \geq 0 \qquad \forall w \in \Omega_0, \tag{7.6}$$

$$\langle \bar{p}^n - p^n - \alpha_n g(v^n, v^n), p - \bar{p}^n \rangle \geq 0 \qquad \forall p \geq 0, \tag{7.7}$$

$$\langle p^{n+1} - p^n - \alpha_n g(v^{n+1}, v^{n+1}), p - p^{n+1} \rangle \geq 0 \qquad \forall p \geq 0. \tag{7.8}$$

Of course, the iteration formulas (7.1) are equivalent to (7.6) – (7.8).

Let us prove the monotone (in norm) convergence of method (7.1) to an equilibrium solution to problem (1.1).

Theorem 2. *Let the solution set of problem* (1.1) *be nonempty,* $F(v)$ *be a monotone operator,* $g(v,w)$ *be a symmetric vector function differentiable with respect to* w *for any* v*, its restriction* $g(v,w)|_{v=w}$ *to the square's diagonal be a convex function, and* $\Omega \subseteq \mathbb{R}^n$ *be a convex closed set.Then, the sequence* v^n *constructed by method* (7.1)*, with* α_n *determined by* (7.2) *or* (7.3)*, converges monotonically in norm to an equilibrium solution to problem* (1.1)*; i.e.,* $v^n \to v^* \in \Omega^*$ *as* $n \to \infty$.

Proof. Setting $w = v^*$ in (7.6) yields

$$\langle v^{n+1} - v^n + \alpha_n (F(v^{n+1}) + \nabla_w^\top g(v^{n+1}, v^{n+1})\bar{p}^n, v^* - v^{n+1}\rangle \geq 0. \tag{7.9}$$

As in (6.6), the last term in (7.9) is transformed as follows:

$$\begin{aligned} &\langle v^{n+1} - v^n, v^* - v^{n+1}\rangle + \alpha_n \langle F(v^{n+1}), v^* - v^{n+1}\rangle + \\ &+\frac{\alpha_n}{2}\langle \bar{p}^n, g(v^*,v^*) - g(v^{n+1}, v^{n+1}\rangle \geq 0. \end{aligned} \tag{7.10}$$

Summing (6.8) and (7.10) gives

$$\begin{aligned} &\langle v^{n+1} - v^n, v^* - v^{n+1}\rangle + \alpha_n \langle F(v^{n+1}) - F(v^*), v^* -^{n+1}\rangle + \\ &+\frac{\alpha_n}{2}\langle \bar{p}^n - p^*, g(v^*,v^*) - g(v^{n+1}, v^{n+1})\rangle \geq 0. \end{aligned} \tag{7.11}$$

Consider inequalities (7.7) and (7.8). Setting $p = p^*$ in (7.8), we have

$$\langle p^{n+1} - p^n, p^* - p^{n+1}\rangle - \alpha_n \langle g(v^{n+1}, v^{n+1}), p^* - p^{n+1}\rangle \geq 0 \tag{7.12}$$

Setting $p = p^{n+1}$ in (7.7) yields

$$\begin{aligned} &\langle \bar{p}^n - p^n, p^{n+1} - \bar{p}^n\rangle + \alpha_n \langle g(v^{n+1}, v^{n+1}) - g(v^n, v^n), p^{n+1} - \bar{p}^n\rangle - \\ &-\alpha_n \langle g(^{n+1}, v^{n+1}), p^{n+1} - \bar{p}^n\rangle \geq 0, \end{aligned} \tag{7.13}$$

We estimate the second term in (7.13) by means of (7.5) and next summing (7.12) and (7.13) to obtain

$$\begin{aligned} &\langle p^{n+1} - p^n, p^* - p^{n+1}\rangle + \langle \bar{p}^n - p^n, p^{n+1} - \bar{p}^n\rangle + \\ &+\alpha_n^2 |g(v^{n+1}, v^{n+1}) - g(v^n, v^n)|^2 - \alpha_n \langle g(v^{n+1}, v^{n+1}), p^* - \bar{p}^n\rangle \geq 0. \end{aligned}$$

By using the obvious relations $\langle \bar{p}^n, g(v^*,v^*)\rangle \leq 0$ and $\langle p^*, g(v^*,v^*)\rangle = 0$, this inequality is represented as

$$\begin{aligned} &\frac{1}{2}\langle \; p^{n+1} - p^n, p^* - p^{n+1}\rangle + \frac{1}{2}\langle \bar{p}^n - p^n, p^{n+1} - \bar{p}^n\rangle + \\ &+\frac{\alpha_n^2}{2}|g(v^{n+1}, v^{n+1} - g(v^n, v^n)|^2 + \frac{\alpha_n}{2}\langle g(v^*,v^*) - g(v^{n+1}, v^{n+1}), p^* - \bar{p}^n\rangle \geq 0. \end{aligned} \tag{7.14}$$

Since the operator $F(v)$ is monotone, the summation of (7.11) and (7.14) gives

$$\begin{aligned} &\langle v^{n+1} - v^n, v^* - v^{n+1}\rangle + \frac{1}{2}\langle p^{n+1} - p^n, p^* - p^{n+1}\rangle + \\ &+\frac{1}{2}\langle \bar{p}^n - p^n, p^{n+1} - \bar{p}^n\rangle + \frac{\alpha_n^2}{2}|g(v^{n+1}, v^{n+1} - g(v^n, v^n)|^2 \geq 0. \end{aligned} \tag{7.15}$$

Using (6.11), we decompose the first three scalar products into a sum of squares:

$$
\begin{aligned}
&|v^{n+1}-v^*|^2+\frac{1}{2}|p^{n+1}-p^*|^2+|v^{n+1}-v^n|^2-\frac{\alpha_n^2}{2}|g(v^{n+1},v^{n+1})-g(v^n,v^n)|^2+\\
&+\frac{1}{2}\left|p^{n+1}-\bar{p}^n\right|^2+\frac{1}{2}\left|\bar{p}^n-p^n\right|^2\le\left|v^n-v^*\right|^2+\frac{1}{2}\left|p^n-p^*\right|^2.
\end{aligned}
\tag{7.16}
$$

In view of

$$
\frac{1}{2}|p^{n+1}-p^n|^2\le|p^{n+1}-\bar{p}^n|^2+|\bar{p}^n-p^n|^2, \tag{7.17}
$$

and in view of (7.3), we can represent (7.16) as

$$
\begin{aligned}
&\left|v^{n+1}-v^*\right|^2+\frac{1}{2}\left|p^{n+1}-p^*\right|^2+\varepsilon\left|v^{n+1}-v^n\right|^2+\frac{1}{4}\left|p^{n+1}-p^n\right|^2\le\\
&\le\left|v^n-v^*\right|^2+\frac{1}{2}\left|p^n-p^*\right|^2.
\end{aligned}
\tag{7.18}
$$

However, if the steplength α_n in (7.1) is determined by (7.2), then the fourth term in (7.16) is estimated using (7.4), and we have

$$
\begin{aligned}
&|v^{n+1}-v^*|^2+\frac{1}{2}|p^{n+1}-p^*|^2+(1-(\alpha_n^2/2)|g|^2)|v^{n+1}-v^n|^2+\\
+\;&\frac{1}{4}|p^{n+1}-p^n|^2\le|v^n-v^*|^2+\frac{1}{2}|p^n-p^*|^2.
\end{aligned}
$$

Since $1-(\alpha_n^2/2)|g|^2\ge\varepsilon$, this inequality has the form of (7.18). Thus, regardless of the method for determining α_n, we arrive at (7.18), which is entirely analogous to (6.12). Thus, the proof of Theorem 2 can be completed following that of Theorem 1. The theorem is proved. □

The above proof can also be extended to the case when the method is applied under disturbance.

8. A PREDICTION-TYPE GRADIENT METHOD WITH RESPECT TO PRIMAL AND DUAL VARIABLES

In the previous sections, we analyzed the so-called implicit iterative schemes, i.e., schemes whose right- and left-hand sides involve the variables of the auxiliary variational inequalities solved at every iteration step. Thus, at every iteration step, one has to solve regularized intermediate variational inequalities or systems of variational inequalities, but with no coupled constraints. These inequalities also represent rather difficult problems. For this reason, the question arises as to whether the situation can be simplified so that every iteration step involves only one or two auxiliary problems of optimizing a strongly convex function on a simple set (projection problems) instead of involving rather difficult variational inequalities. The answer is affirmative. Consider one of the possible gradient iterative schemes with a predictive step with respect to both primal and dual variables. Let (v^0,p^0) be the initial approximation. The next approximation is calculated using the recurrence relations (see [22])

$$
\begin{aligned}
\bar{p}^n &= \pi_+(p^n+\alpha_n g(v^n,v^n)),\\
\bar{v}^n &= \pi_{\Omega_0}(v^n-\alpha_n(F(v^n)+\nabla_w^\top g(v^n,v^n)\bar{p}^n)),\\
p^{n+1} &= pi_+(p^n+\alpha_n g(\bar{v}^n,\bar{v}^n)),\\
v^{n+1} &= \pi_{\Omega_0}(v^n-\alpha_n(F(\bar{v}^n)+\nabla_w^\top g(\bar{v}^n,\bar{v}^n)\bar{p}^n)).
\end{aligned}
\tag{8.1}
$$

The step length α_n in (8.1) is determined by either the condition

$$0 < \varepsilon \le \alpha_n < 1/\sqrt{2(|F|^2 + C^2|\nabla|^2) + \frac{1}{2}|g|^2}, \quad \varepsilon > 0, \tag{8.2}$$

where $|F|$, $|\nabla|$, C, and $|g|$ are the constants defined by (8.4) and (8.6), or the condition

$$\begin{aligned} \alpha_n^2(|F(\bar{v}^n) - F(v^n) \ &+ \ (\nabla_w^\top g(\bar{v}^n, \bar{v}^n) - \nabla_w^\top g(v^n, v^n))\bar{p}^n|^2 + \\ &+ \ (1/2)|g(\bar{v}^n, \bar{v}^n) - g(v^n, v^n)|^2) \le (1-\varepsilon)|\bar{v}^n - v^n|^2. \end{aligned} \tag{8.3}$$

The step length in (8.2) and (8.3) is chosen following the same scheme as in (7.2) and (7.3). The difference between $\bar{v}^n$ and v^{n+1} and between $\bar{p}^n$ and p^{n+1} in (8.1) are estimated as

$$\begin{aligned} &|\bar{p}^n - p^{n+1}| \le \alpha_n|g(v^n, v^n) - g(\bar{v}^n, \bar{v}^n)|, \\ &|\bar{v}^n - v^{n+1}| \le \alpha_n|F(v^n) - F(\bar{v}^n) + (\nabla_w^\top g(v^n, v^n) - \nabla_w^\top g(\bar{v}^n, \bar{v}^n))\bar{p}^n|. \end{aligned} \tag{8.4}$$

The methods of determining α_n by (8.2) and (8.3) are substantiated as follows. Assume that $g(v, w)$, $F(v)$, and $\nabla_w^\top g(v, v)$ satisfy the Lipschitz conditions

$$|g(v+h, v+h) - g(v, v)| \le |g||h| \tag{8.5}$$

for all $v \in \Omega$ and $h \in R^n$ (where $|g|$ is a constant) and

$$|F(v+h) - F(v)| \le |F||h|, \qquad |\nabla_w^\top g(v+h, v+h) - \nabla_w^\top g(v, v))| \le |\nabla||h|, \tag{8.6}$$

for all $v \in \Omega$ and $h \in R^n$, where $|F|$ and $|\nabla|$ are constants; moreover, let $|\bar{p}^n| \le C$.

From (8.5) and (8.6), we have

$$|F(\bar{v}^n) - F(v^n) + (\nabla_w^\top g(\bar{v}^n, \bar{v}^n) - \nabla_w^\top g(v^n, v^n))\bar{p}^n| \le (|F| + |\bar{p}^n||\nabla|)|\bar{v}^n - v^n|,$$

$$|g(\bar{v}^n, \bar{v}^n) - g(v^n, v^n)| \le |g||\bar{v}^n - v^n|.$$

Since $|\bar{p}^n| \le C$,

$$\begin{aligned} &|F(\bar{v}^n) - F(v^n) + (\nabla_w^\top g(\bar{v}^n, \bar{v}^n) - \nabla_w^\top g(v^n, v^n))\bar{p}^n|^2 + \\ &+(1/2)|g(\bar{v}^n, \bar{v}^n) - g(v^n, v^n)|^2 \le \{|F| + `|\nabla|)^2 + (1/2)|g|^2\}|\bar{v}^n - v^n|^2. \end{aligned} \tag{8.7}$$

Obviously, if $(|F| + `|\nabla|)^2 + (1/2)|g|^2 \le (1-\varepsilon)/\alpha_n^2$, i.e.,

$$\alpha_n^2 \le \frac{1-\varepsilon}{(|F| + C|\nabla|)^2 + (1/2)|g|^2},$$

then there exists α_n satisfying (8.3).

Method (8.1) can be represented as variational inequalities. By the definition of the projection operator, the first and third equations in (8.1) are written as

$$\langle \bar{p}^n - p^n - \alpha_n g(v^n, v^n), p - \bar{p}^n \rangle \ge 0 \ \ \forall p \ge 0, \tag{8.8}$$

and

$$\langle p^{n+1} - p^n - \alpha_n g(\bar{u}^n, \bar{u}^n), p - p^{n+1} \rangle \ge 0 \ \ \forall p \ge 0. \tag{8.9}$$

The second and fourth equations are represented as

$$\langle \bar{v}^n - v^n + \alpha_n(F(v^n) + \nabla_w^\top g(v^n, v^n)\bar{p}^n), w - \bar{v}^n \rangle \ge 0 \ \ \forall w \in \Omega_0, \tag{8.10}$$

and

$$\langle v^{n+1} - v^n + \alpha_n(F(\bar{v}^n) + \nabla_w^\top g(\bar{v}^n, \bar{v}^n)\bar{p}^n), w - v^{n+1}\rangle \geq 0 \quad \forall w \in \Omega_0, \tag{8.11}$$

Let us show that method (8.1) converges monotonically in norm to an equilibrium solution.

Theorem 3. *Let the solution set of problem* (1.1) *be nonempty, $F(v)$ be a monotone operator, $g(v,w)$ be a symmetric vector function differentiable and convex with respect to w for any v, its restriction $g(v,w)|_{v=w}$ to the square's diagonal be a convex function, $\Omega \subseteq I\!R^n$ be a convex closed set, and $|p^n| \leq C$ for all n. Then, the sequence $v^n n$ constructed by method* (8.1) *with α_n determined by* (8.2) *or* (8.3) *converges monotonically in norm to an equilibrium solution to problem* (1.1)*; i.e., $v^n \to v^* \in \Omega^*$ as $n \to \infty$.*

Proof. Setting $w = v^*$ in (8.11) yields

$$\langle v^{n+1} - v^n, v^* - v^{n+1}\rangle + \alpha_n\langle F(\bar{v}^n), v^* - v^{n+1}\rangle + \alpha_n\langle \nabla_w^\top g(\bar{v}^n, \bar{v}^n)\bar{p}^n, v^* - v^{n+1}\rangle \geq 0. \tag{8.12}$$

Setting $w = v^{n+1}$ in (8.10), we have

$$\langle \bar{v}^n - v^n + \alpha_n(F(v^n) + \nabla_w^\top g(v^n, v^n)\bar{p}^n), v^{n+1} - \bar{v}^n\rangle \geq 0.$$

It follows that

$$\begin{aligned}&\langle \bar{v}^n - v^n, v^{n+1} - \bar{v}^n\rangle + \alpha_n\langle F(\bar{v}^n), v^{n+1} - \bar{v}^n\rangle -\\ &-\alpha_n\langle F(\bar{v}^n) - F(v^n), v^{n+1} - \bar{v}^n\rangle + \alpha_n\langle \nabla_w^\top g(\bar{v}^n, \bar{v}^n)\bar{p}^n, v^{n+1} - \bar{v}^n\rangle -\\ &-\alpha_n\langle (\nabla_w^\top g(\bar{v}^n, \bar{v}^n) - \nabla_w^\top g(v^n, v^n))\bar{p}^n, v^{n+1} - \bar{v}^n\rangle \geq 0,\end{aligned}$$

or, in view of (8.4),

$$\begin{aligned}&\langle \bar{v}^n - v^n, v^{n+1} - \bar{v}^n\rangle + \alpha_n\langle F(\bar{v}^n), v^{n+1} - \bar{v}^n\rangle +\\ &+\alpha_n\langle \nabla_w^\top g(\bar{v}^n, \bar{v}^n)\bar{p}^n, v^{n+1} - \bar{v}^n\rangle +\\ &+\alpha_n^2|F(\bar{v}^n) - F(v^n) + (\nabla_w^\top g(\bar{v}^n, \bar{v}^n) - \nabla_w^\top g(v^n, v^n))\bar{p}^n|^2 \geq 0.\end{aligned} \tag{8.13}$$

Summing (8.12) and (8.13) produces

$$\begin{aligned}&\langle v^{n+1} - v^n, v^* - v^{n+1}\rangle + \langle \bar{v}^n - v^n, v^{n+1} - \bar{v}^n\rangle + \alpha_n\langle F(\bar{v}^n), v^* - \bar{v}^n\rangle +\\ &+\alpha_n\langle \bar{p}^n, \nabla_w g(\bar{v}^n, \bar{v}^n)(v^* - \bar{v}^n)\rangle + \alpha_n^2|F(\bar{v}^n) - F(v^n) +\\ &+(\nabla_w^\top g(\bar{v}^n, \bar{v}^n) - \nabla_w^\top g(v^n, v^n))\bar{p}^n|^2 \geq 0.\end{aligned} \tag{8.14}$$

As in (6.6), the fourth term in (8.14) is transformed as follows:

$$\langle \bar{p}^n, \nabla_w g(\bar{v}^n, \bar{v}^n)(v^* - \bar{v}^n)\rangle = \frac{1}{2}\langle \bar{p}^n, \nabla g(\bar{v}^n, \bar{v}^n)(v^* - \bar{v}^n)\rangle \leq \frac{1}{2}\langle \bar{p}^n, g(v^*, v^*) - g(\bar{v}^n, \bar{v}^n)\rangle,$$

then,

$$\begin{aligned}&\langle v^{n+1} - v^n, v^* - v^{n+1}\rangle + \langle \bar{v}^n - v^n, v^{n+1} - \bar{v}^n\rangle + \alpha_n\langle F(\bar{v}^n), v^* - \bar{v}^n\rangle +\\ &+(\alpha_n/2)\langle \bar{p}^n, g(v^*, v^*) - g(\bar{v}^n, \bar{v}^n)\rangle +\\ &+\alpha_n^2|F(\bar{v}^n) - F(v^n) + (\nabla_w^\top g(\bar{v}^n, \bar{v}^n) - \nabla_w^\top g(v^n, v^n))\bar{p}^n|^2 \geq 0.\end{aligned}$$

Setting $w = \bar{v}^n$ in the first inequality of (5.5), we have

$$\langle F(v^*), \bar{v}^n - v^*\rangle + \frac{1}{2}\langle p^*, g(\bar{v}^n, \bar{v}^n) - g(v^*, v^*)\rangle \geq 0.$$

Summing the last two inequalities yields

$$\begin{aligned}&\langle v^{n+1}-v^n, v^*-v^{n+1}\rangle+\langle \bar v^n-v^n, v^{n+1}-\bar v^n\rangle+\alpha_n\langle F(\bar v^n)-F(v^*), v^*-\bar v^n\rangle+\\ &+\frac{\alpha_n}{2}\langle \bar p^n-p^*, g(v^*,v^*)-g(\bar v^n,\bar v^n)\rangle+\alpha_n^2|F(\bar v^n)-F(v^n)+\\ &+(\nabla_w^\top g(\bar v^n,\bar v^n)-\nabla_w^\top g(v^n,v^n))\bar p^n|^2\geq 0.\end{aligned}\tag{8.15}$$

Consider inequalities (8.8) and (8.9). Setting $p=p^*$ in (8.9), we have

$$\langle p^{n+1}-p^n, p^*-p^{n+1}\rangle-\alpha\langle g(\bar v^n,\bar v^n), p^*-p^{n+1}\rangle\geq 0\tag{8.16}$$

Setting $p=p^{n+1}$ in (8.8) yields

$$\begin{aligned}&\langle \bar p^n-p^n, p^{n+1}-\bar p^n\rangle+\alpha_n\langle g(\bar v^n,\bar v^n)-g(v^n,v^n), p^{n+1}-\bar p^n\rangle-\\ &-\alpha_n\langle g(\bar v^n,\bar v^n), p^{n+1}-\bar p^n\rangle\geq 0.\end{aligned}\tag{8.17}$$

Estimating the second term in (8.17) by means of (8.4) and summing (8.16) and (8.17), we obtain

$$\begin{aligned}&\langle p^{n+1}-p^n, p^*-p^{n+1}\rangle+\langle \bar p^n-p^n, p^{n+1}-\bar p^n\rangle+\\ &+\alpha_n^2|g(\bar v^n,\bar v^n)-g(v^n,v^n)|^2-\alpha_n\langle g(\bar v^n,\bar v^n), p^*-\bar p^n\rangle\geq 0.\end{aligned}$$

By using the relations $\langle \bar p^n, g(v^*,v^*)\rangle\leq 0$ and $\langle p^*, g(v^*,v^*)\rangle=0$, this inequality can be rewritten as

$$\begin{aligned}&\frac{1}{2}\langle p^{n+1}-p^n, p^*-p^{n+1}\rangle+\frac{1}{2}\langle \bar p^n-p^n, p^{n+1}-\bar p^n\rangle+\\ &+\frac{\alpha_n^2}{2}|g(\bar v^n,\bar v^n)-g(v^n,v^n)|^2+\frac{\alpha_n}{2}\langle g(v^*,v^*)-g(\bar v^n,\bar v^n), p^*-\bar p^n\rangle\geq 0.\end{aligned}\tag{8.18}$$

Summing (8.15) and (8.18) and taking into account the monotonicity of $F(v)$ gives

$$\begin{aligned}&\langle v^{n+1}-v^n, v^*-v^{n+1}\rangle+\langle \bar v^n-v^n, v^{n+1}-\bar v^n\rangle+\\ &+(1/2)\langle p^{n+1}-p^n, p^*-p^{n+1}\rangle+(1/2)\langle \bar p^n-p^n, p^{n+1}-\bar p^n\rangle+\\ &+\alpha_n^2(|F(\bar v^n)-F(v^n)+(\nabla_w^\top g(\bar v^n,\bar v^n)-\nabla_w^\top g(v^n,v^n))\bar p^n|^2+\\ &+(1/2)|g(\bar v^n,\bar v^n)-g(v^n,v^n)|^2)\geq 0.\end{aligned}$$

Using (6.11), we decompose the first four scalar products into a sum of squares to obtain

$$\begin{aligned}&|v^{n+1}-v^*|^2+(1/2)|p^{n+1}-p^*|^2+|v^{n+1}-\bar v^n|^2+|\bar v^n-v^n|^2+\\ &+(1/2)|p^{n+1}-\bar p^n|^2+(1/2)|\bar p^n-p^n|^2+\\ &+\alpha_n^2(|F(\bar v^n)-F(v^n)+(\nabla_w^\top g(\bar v^n,\bar v^n)-\nabla_w^\top g(v^n,v^n))\bar p^n|^2+\\ &+(1/2)|g(\bar v^n,\bar v^n)-g(v^n,v^n)|^2)\leq |v^n-v^*|^2+(1/2)|p^n-p^*|^2.\end{aligned}$$

By virtue of (7.17), the last inequality can be represented as

$$\begin{aligned}&|v^{n+1}-v^*|^2+(1/2)|p^{n+1}-p^*|^2+(1/4)|p^{n+1}-p^n|^2+|v^{n+1}-\bar v^n|^2+\\ &+|\bar v^n-v^n|^2-\alpha_n^2(|F(\bar v^n)-F(v^n)+(\nabla_w^\top g(\bar v^n,\bar v^n)-\nabla_w^\top g(v^n,v^n))\bar p^n|^2+\\ &+(1/2)|g(\bar v^n,\bar v^n)-g(v^n,v^n)|^2)\leq |v^n-v^*|^2+(1/2)|p^n-p^*|^2.\end{aligned}\tag{8.19}$$

Estimating the sum of the fifth and sixth terms in (8.19) with the use of (8.3), we have

$$\begin{aligned}|v^{n+1}-v^*|^2 &+ \frac{1}{2}|p^{n+1}-p^*|^2+\frac{1}{4}|p^{n+1}-p^n|^2+|v^{n+1}-\bar v^n|^2+\varepsilon|\bar v^n-v^n|^2\leq\\ &\leq |v^n-v^*|^2+\frac{1}{2}|p^n-p^*|^2.\end{aligned}\tag{8.20}$$

However, if the step length α_n in (8.1) is determined by (8.2), the sixth term in (8.19) is estimated by using (8.5) and (8.6), in conjunction with $\langle x, y\rangle \le |x|^2 + |y|^2$, and we have

$$|v^{n+1} - v^*|^2 + \frac{1}{2}|p^{n+1} - p^*|^2 + \frac{1}{4}|p^{n+1} - p^n|^2 + |v^{n+1} - \bar{v}^n|^2 + \\ +(1 - \alpha_n^2(2(|F|^2 + C^2|\nabla|^2) + \frac{1}{2}|g|^2)|\bar{v}^n - v^n|^2 \le |v^n - v^*|^2 + \frac{1}{2}|p^n - p^*|^2. \qquad (8.21)$$

Since $1 - \alpha_n^2(2(|F|^2 + C^2|\nabla|^2) + (1/2)|g|^2) \ge \varepsilon$, we conclude that (8.21) has the form of (8.20). Thus, regardless of the method for determining α_n, we arrive at (8.20), which is entirely analogous to (6.12). Therefore, the proof of Theorem 3 can be completed following that of Theorem 1. The theorem is proved. □

The above proof can also be extended to situation under disturbance.

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