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EVOLVING SYSTEMS

EQUILIBRIUM PROGRAMMING: GRADIENT METHODS[1]

A.S. Antipin

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The equilibrium programming problem is formulated and its relationship with game formulation is discussed. A forecast method for computing the equilibrium solution is designed and its convergence is proved. The economic interpretation of the initial equilibrium problem and its solution are examined.

1. INTRODUCTION

It is now generally recognized that a well-founded theory underlies the methods of solving optimization problems, whereas there is no such theory to authenticate the methods of solving game problems, e.g., saddle problems, n-person games under Nash equilibrium, inverse optimization problems, and economic equilibrium models.

The need for developing a theory of methods of solving equilibrium problems is obvious, because precisely these are the problems that describe the fine points underlying the ideas of compromise between partially (or fully) conflicting factors and interests in terms of models. The methods of solving equilibrium problems are interpreted as mechanisms for matching conflicting factors. In this paper, we examine an equilibrium programming problem whose solution is a fixed point, and design a fairly general approach to computing this point. The approach is based on an inequality, which compensates for the lack of monotonicity an equilibrium problems, and a controlled gradient descent in the form of a forecast, which compensates for the lack of the potentiality property.

2. FORMULATION OF THE PROBLEM

The equilibrium programming problem consists in finding a fixed point $v^* \in \Omega^*$ that satisfies the extremal inclusion under functional constraints

$$v^* \in \operatorname{Argmin}\{\Phi(v^*, w) \mid g(w) \leq 0, \;\; w \in \Omega\}. \tag{2.1}$$

The function $\Phi(v, w)$ is defined on the product space $\mathbb{R}^n \times \mathbb{R}^n$ and $\Omega \subset \mathbb{R}^n$ is a closed convex set. We assume that the function $\Phi(v, w)$ is convex relative to the variable $w \in \Omega$ for every $v \in \Omega$. The vector function $g(w)$ is of dimension m, and each of its components is convex for all $w \in \mathbb{R}^n$. The variable $v \in \Omega$ in (2.1) plays the part of a parameter and $w \in \Omega$ is the optimization variable. We also assume that the extremal (marginal) mapping $w(v) \equiv \operatorname{argmin}\{\Phi(v, w) \mid g(w) \leq 0, \;\; w \in \Omega\}$ is defined for all $v \in \Omega$ and the set of solutions $\Omega^* \subset \Omega$ of the initial problem is not empty. The last assumption, according to the Caccutani theorem, always holds if Ω is a convex compact and $\Phi(v, w)$ is semicontinuous in v from below and convex in w [1].

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Many known problems of analysis, for example, the saddle problems or the n-person games under Nash equilibrium in general formulation, can be reduced to the form (2.1). Indeed, let the inequalities

$$L(x^*, y) \leq L(x^*, p^*) \leq L(z, p^*) \qquad \forall z \in Q \subseteq \mathbb{R}^n \quad \forall y \in P \subseteq \mathbb{R}^m \tag{2.2}$$

define a saddle point, where $L(z, y)$ is a convex-concave function, $Q = \{z \mid g_1(z) \leq \leq 0,\ z \in Q_1\}$, $P = \{y \mid g_2(y) \leq 0,\ y \in P_1\}$, and $g_1(z)$ and $g_2(y)$ are convex vector functions. Let $w = (z, y)$, $v = (x, p)$, $v^* = (x^*, p^*)$, $g(w) = (g_1(z), g_2(z))$, and let $\Phi(v, w) = L(z, p) - L(x, y)$ be a normalized function. In this notation, problem (2.2) can always be expressed in the equivalent form (2.1) [2, 3].

The more general n-person game under Nash equilibrium can also be reduced to the form (2.1). Indeed, the payoff function of the ith player $f_i(x_i, x_{-i})$, $i \in I$, depends on the strategies $x_i \in X_i$, where $X_i = (x_i)_{i\in I}$, of the ith player as well as on the strategies $x_{-i} = (x_j)_{j\in I\setminus i}$ of other players. The equilibrium of the n-person game is the solution of the system of extremal inclusions

$$x_i^* \in \operatorname{Argmin}\{f_i(x_i, x_{-i}^*) \mid g_i(x_i) \leq 0,\ \ x_i \in X_i\}. \tag{2.3}$$

Let us introduce the normalized function

$$\Phi(v, w) = \sum_{i=1}^{n} f_i(x_i, x_{-i}),$$

where $v = (x_{-i})$, $w = (x_i)$, $g(w) = (g_i(x_i))$, $i = 1, \ldots, n$, $\Omega = X_1 \times X_2 \times \ldots \times X_n$, and $(v, w) = (x_i, x_{-i}) \in \Omega \times \Omega$. Using this function, we can express problem (2.3) as (2.1).

Many inverse optimization problems [4] can also be represented as (2.1). For example, let us consider the inverse convex programming problem

$$\begin{gathered} x^* \in \operatorname{Argmin}\{\langle \lambda^*, f(x)\rangle \mid g(x) \leq 0,\ \ x \in Q\}, \\ G(x^*) \leq d. \end{gathered} \tag{2.4}$$

In this problem, it is required to choose nonnegative coefficients of the linear convolution $\lambda = \lambda^*$ such that the optimal solution x^* corresponding to these weights belongs to a predefined convex set. In particular, this set may be a singleton. It is assumed that the functions (2.4) are all convex.

System (2.4) can be represented as a two-person game under Nash equilibrium:

$$\begin{aligned} x^* &\in \operatorname{Argmin}\{\langle \lambda^*, f(x)\rangle \mid g(x) \leq 0,\ \ x \in Q\}, \\ p^* &\in \operatorname{Argmin}\{\langle p, G(x^*) - d\rangle \mid p \geq 0\}. \end{aligned} \tag{2.5}$$

Using the normalized function, we can reduce problem (2.5) to the form (2.1). Thus, the initial inverse optimization problem (2.4) is reduced to computing the fixed point of the extremal mapping (2.1).

3. ANTISYMMETRIC FUNCTIONS

In the sequel, we primarily use two basic inequalities, of which the first — an equivalent definition of the fixed point for (2.1) — is of the form

$$\Phi(v^*) \leq \Phi(v^*, w) \qquad \forall w \in D, \tag{3.1}$$

where $D = \{w \mid g(w) \le 0, \ \ w \in \Omega\}$ is the admissible set. Since $\Phi^* = \inf\{\Phi(w,w) \mid w \in$ $\in D\} \le \Phi(v^*, v^*)$, (3.1) immediately yields the Ky Fan inequality [5]

$$\inf\{\Phi(w,w) \mid w \in D\} \le \Phi(v^*, w). \tag{3.2}$$

This inequality is equivalent to the Caccutani theorem [1] and asserts the existence of a fixed point for problem (2.1).

The second inequality defines an antisymmetric (saddle) property of the fixed point [6, 7] and is of the form

$$\Phi(w, v^*) \le \Phi(w,w) \qquad \forall w \in D. \tag{3.3}$$

Rewriting (3.3) in a more general form as

$$\Phi(w, v^*) \le \sup\{\Phi(w,w) \mid w \in D\} \tag{3.4}$$

and comparing it with Ky Fan inequality (3.2)

$$\inf\{\Phi(w,w) \mid w \in D\} \le \Phi(v^*, w), \tag{3.5}$$

we find that both inequalities can be regarded as the generalized definition of the saddle point for $\sup\{\ldots\} = \inf\{\ldots\} = \Phi(v^*, v^*)$.

The geometric meaning of inequality (3.3) is quite obvious. Taking on the diagonal of the square $\Omega \times \Omega$ two close points with coordinates v^*, v^* and w, w, where $v^* \in \Omega^*$, $w \in \Omega$, we find that these points uniquely define two other points with coordinates w, v^* and v^*, w. These four points define a small square belonging to the square $\Omega \times \Omega$. Now let us consider the Lebesgue set generated by the point w, w:

$$R_{w,w} = \{u_1, u_2 \mid \Phi(u_1, u_2) \le \Phi(w,w), \ \ u_1 \in D, \ \ u_2 \in D\}.$$

By definition, points with coordinates w, w belong to this set. Condition (3.3) implies that the point with coordinates w, v^* also belongs to this set. Thus, (3.3) actually implies that the Lebesgue set (since $\Phi(v, w)$ is convex in w) contains the entire interval $\alpha v^* +$ $+ (1-\alpha) w, \ \ 0 \le \alpha \le 1$.

If the function $\Phi(v, w)$ is differentiable with respect to all variables, inequality (3.3) can be refined. Using the convexity condition

$$\langle \nabla f(x), y - x \rangle \le f(y) - f(x) \le \langle \nabla f(y), y - x \rangle, \tag{3.6}$$

which holds for all x and y belonging to some set, let us transform (3.3) as

$$\begin{aligned} 0 \ &\le \ \Phi(w,w) - \Phi(w, v^*) \le \langle \nabla \Phi_w(w,w), w - v^* \rangle = \\ &= \ \langle \nabla \Phi_v(w,w), 0 \rangle + \langle \nabla \Phi_w(w,w), w - v^* \rangle = \langle \nabla \Phi(w,w), h \rangle, \end{aligned}$$

where $h = (0, w - v^*)$ and $\nabla \Phi(w,w) = (\nabla \Phi_v(v,w), \nabla \Phi_w(v,w))$. Hence, if (3.3) is satisfied for all w belonging to some neighborhood of the equilibrium point v^*, the gradient $\nabla \Phi(v, w)$ makes an acute (more exactly, not obtuse) angle with the vector h for all w belonging to this neighborhood. This condition for the optimization problem takes the form $\langle \nabla \Phi(w), w - v^* \rangle \ge 0 \ \ \forall w \in D$ [8].

Inequality (3.3) has no constructive value, because it contains an unknown vector $v^* \in \Omega^*$. Therefore, let us introduce a class of functions for which conditions (3.3) are always satisfied [7].

Definition 1. *A function* $\Phi(v, w)$ *belonging to the product* $\mathbb{R}^n \times \mathbb{R}^n$ *in the space* $\mathbb{R}^1$ *is said to be antisymmetric on* $\Theta \times \Theta$ *if it satisfies the inequality*

$$\Phi(w, w) - \Phi(w, v) - \Phi(v, w) + \Phi(v, v) \geq 0 \tag{3.7}$$

for all $w \in \Theta$ *and all* $v \in \Theta$.

If an inequality of the type

$$\Phi(w, w) - \Phi(w, v^*) - \Phi(v^*, w) + \Phi(v^*, v^*) \geq 0 \tag{3.8}$$

holds for all w *belonging to some neighborhood of the solution* $v^* \in \Omega^*$*, the function* $\Phi(v, w)$ *is said to be antisymmetric relative to equilibrium.*

In what follows, we assume that the set Θ is either Ω or D.

The class of symmetric functions introduced above is not empty. It is a simple matter to verify [2, 9] that the normalized function $\Phi(v, w) = L(z, p) - L(x, y)$, $w = (z, y)$, $v = (x, p)$, of the saddle problem (2.2) is symmetric.

Condition (3.3) is always satisfied for symmetric functions. Indeed, using (3.1), from (3.8) we obtain (3.3).

Symmetric functions have certain properties [7], which can be regarded as analogs of the conditions governing the gradient monotonicity and nonnegativity of the second derivative of convex functions.

Property 1. *If a function* $\Phi(v, w)$ *is antisymmetric and convex in the second variable, its partial gradient* $\nabla\Phi_w(v, v)$ *is monotonic on the diagonal of the square* $\Theta \times \Theta$*:*

$$\langle \nabla\Phi_w(w, w) - \nabla\Phi_w(v, v), w - v \rangle \geq 0 \qquad \forall w \in \Theta,\ v \in \Theta. \tag{3.9}$$

This inequality can also be derived by estimating the first two and last two terms on the left side of (3.7) with (3.6).

Property 2. *The mixed derivative* $\nabla^2\Phi_{wv}(v, v)$ *of the antisymmetric function* $\Phi(v, w)$ *on the diagonal of the square* $\Theta \times \Theta$ *is nonnegative:*

$$\langle \nabla^2\Phi_{wv}(v, v)h, h \rangle \geq 0 \qquad \forall h \in \mathbb{R}^n. \tag{3.10}$$

We can easily verify that functions $\Phi_1(v, w) = \langle v, w \rangle$, $\Phi_2(v, w) = \dfrac{1}{2}|v + w|^2$, and $\Phi_3(v, w) = v^2 + w^2$ satisfy conditions (3.9) and (3.10).

4. EXAMPLES

The examples given below show that equilibrium problems obeying condition (3.3) are rather diverse in nature. Let us determine the fixed point of the quadratic extremal inclusion

$$v^* \in \text{Argmin}\left\{\frac{1}{2}\langle Nw, w\rangle + \langle Mv^* + m, w\rangle \mid w \in \Omega\right\}, \tag{4.1}$$

where N and M are nonnegative matrices, i.e., $\langle Nv, v\rangle \geq 0$ and $\langle Mv, v\rangle \geq 0$ for all $v \in \mathbb{R}^n$. Assuming that the matrix N is symmetric, let us consider

$$\begin{aligned} & \Phi(w,w) - \Phi(w,v) - \Phi(v,w) + \Phi(v,v) = \\ = \;& \frac{1}{2} - \langle Nw, w\rangle + \langle Mw, w\rangle + \langle m, w\rangle - \frac{1}{2}\langle Nv, v\rangle - \langle Mw, v\rangle - \langle m, v\rangle - \\ - \;& \frac{1}{2}\langle Nw, w\rangle - \langle Mv, w\rangle - \langle m, w\rangle + \frac{1}{2}\langle Nv, v\rangle + \langle Mv, v\rangle + \langle m, v\rangle = \\ = \;& \langle M(w-v), w-v\rangle \geq 0 \qquad \forall w \in \Omega, \qquad \forall v \in \Omega. \end{aligned} \tag{4.2}$$

Hence, if the matrix M is nonnegative, then condition (3.7) is satisfied for a function $\Phi(w, v)$ belonging to (4.1).

We now construct the matrices N and M. Let N — (2×2) be a unit matrix and let M — (2×2) be an antisymmetric matrix with unit components $m = (a, a)$, $v = (x, p)$, and $w = (z, y)$. Then the function $\Phi(v, w)$ takes the form

$$\Phi(v,w) = (z,y)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} z \\ y \end{pmatrix} + (x,p)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} z \\ y \end{pmatrix} - (a,a)\begin{pmatrix} z \\ y \end{pmatrix}, \tag{4.3}$$

or $\Phi(v, w) = z^2 + y^2 - xy + pz - az - ay$. The function $\Phi(v, w)$ is a separable structure for the minimization variable w. Therefore, minimization of the function $\Phi(v, w)$ on Ω breaks down into two independent subproblems with the objective functions

$$\begin{aligned} f_1(z,p) &= z(z + p - a), \\ f_2(x,y) &= y(-x + y - a). \end{aligned} \tag{4.4}$$

The vector $v = (x, p)$ here plays the part of a parameter.

Taking $v = v^*$, let us formulate a two-person game under Nash equilibrium:

$$\begin{aligned} z^* &= \text{argmin}\{f_1(z, y^*) = z(z + y^* - a) \mid z \in \mathbb{R}^1\}, \\ y^* &= \text{argmin}\{f_2(z^*, y) = y(-z^* + y - a) \mid y \in \mathbb{R}^1\}. \end{aligned} \tag{4.5}$$

For game (4.5), function (4.3) is normalized. Clearly, this is a zero-sum game and condition (3.7) is satisfied, since $\langle Mh, h\rangle \geq 0$ for all $h \in \mathbb{R}^n$.

Let us consider one more example. Assuming that the matrix M is symmetric but not positive-definite, i.e

$$\Phi(v,w) = (z,y)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} z \\ y \end{pmatrix} + (x,p)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} z \\ y \end{pmatrix} - (a,a)\begin{pmatrix} z \\ y \end{pmatrix}, \tag{4.6}$$

we obtain $\Phi(v, w) = z^2 + y^2 + xy + pz - az - ay$. Reasoning as in (4.3) and (4.4), we find that function (4.6) is normalized for the game

$$\begin{aligned} z^* &= \text{argmin}\{f_1(z, y^*) = z(z + y^* - a) \mid z \in [0, a]\}, \\ y^* &= \text{argmin}\{f_2(z^*, y) = y(z^* + y - a) \mid y \in [0, a]\}, \end{aligned} \tag{4.7}$$

where $a > 0$.

This problem is the basic model representing the behavior of two monopolists who manufacture the same product and compete in the same market area (Cournot duopoly [10]). Let z and y be the turnout of the first and second monopolist, respectively. If the second monopolist produces y^* units, the first monopolist in ibis case dumps $z^* = \frac{1}{2}(a - y^*)$ units that minimize his functional costs $f_1(z, y^*) = z(z + y^* - a)$. A similar strategy $y^* = \frac{1}{2}(a - z^*)$ is adopted by the second monopolist if he knows that the first monopolist dumps z^* units. The fixed point of the duopoly equilibrium is the pair $z^* = \frac{1}{3}a$, $y^* = \frac{1}{3}a$. The outlays of the monopolists are $-\frac{a^2}{9}$. Condition (3.3) holds for this problem. We now verify this fact.

Let $w = v^* = (x^*, p^*)$ and $v = w = (z, y)$. Then (3.3) can be expressed as

$$(x^*, p^*)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x^* \\ p^* \end{pmatrix} + (z, y)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x^* \\ p^* \end{pmatrix} - (a, a)\begin{pmatrix} x^* \\ p^* \end{pmatrix} \leq$$
$$\leq \quad (z, y)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} z \\ y \end{pmatrix} + (z, y)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} z \\ y \end{pmatrix} - (a, a)\begin{pmatrix} z \\ y \end{pmatrix}.$$

Hence, $(x^*)^2 + (p^*)^2 + zp^* + yx^* - ax^* - ap^* \leq z^2 + y^2 + 2zy - az - ay$. Since $x^* = \frac{1}{3}a$ and $p^* = \frac{1}{3}a$, we find that $-\frac{4}{9}a^2 + \frac{1}{3}a(z + y) \leq (z + y)^2 - a(z + y)$. Finally, $0 \leq \left((z + y) - \frac{2}{3}a\right)^2$. Therefore, condition (3.3) is satisfied for game (4.7).

5. FORECAST-TYPE GRADIENT METHOD

We now examine the methods of solving problem (2.1). First, we make a remark. Representing inequalities (3.1) and (3.3) as

$$\Phi(w, v^*) - \Phi(w, w) \leq \Phi(v^*, v^*) - \Phi(v^*, v^*) \leq \Phi(v^*, w) - \Phi(v^*, v^*) \qquad \forall w \in D, \quad (5.1)$$

and introducing the function $\Psi(v, w) = \Phi(v, w) - \Phi(v, v)$, let us express system of inequalities (5.1) as

$$\Psi(w, v^*) \leq \Psi(v^*, v^*) \leq \Psi(v^*, w) \qquad \forall w \in D. \quad (5.2)$$

From this system of inequalities, we find that the point v^*, v^* is a saddle point of the function. A question arises here: is it possible to use the function $\Psi(v, w)$ for computing the saddle point v^*, v^*, and, thereby, the fixed point v^*. The answer is obvious: in general, the function $\Psi(v, w)$ cannot be used, because it is concave in v, though it is convex in w. The method for computing the saddle point of the function $\Psi(v, w)$ [11] must necessarily contain a procedure for shifting v as well as w. Consequently, the method will not be satisfactory in convergence, because the function is not concave relative to v. Our methods incorporate a technique in which equilibrium (in particular, saddle) is approached through iterations on one variable w; moreover, the process monotonically converges (in the norm of the space) to equilibrium in the degenerate case.

Following the familiar logic of convex programming, let us introduce a Lagrange function for problem (2.1)

$$L(v^*, w, p) = \Phi(v^*, w) + \langle p, g(w) \rangle, \qquad w \in \Omega, \quad p \geq 0.$$

If the functional constraints are regular (for instance, the Slater condition holds), this problem can be reduced to computing the saddle point of the Lagrange function $L(v^*, w, p)$, i.e.,

$$\Phi(v^*, v^*) + \langle p, g(v^*) \rangle \leq \Phi(v^*, v^*) + \langle p^*, g(v^*) \rangle \leq \Phi(v^*, w) + \langle p^*, g(w) \rangle \tag{5.3}$$

for all $w \in \Omega$ and $p \geq 0$.

If the functions $\Phi(v, w)$ and $g(w)$ are differentiable, then (5.3) can expressed in equivalent form as

$$\begin{aligned} v^* &= \pi_\Omega(v^* - \alpha \nabla L_w(v^*, v^*, p^*)), \\ p^* &= \pi_+(p^* + \alpha g(v^*)), \end{aligned} \tag{5.4}$$

where $\pi_+(\cdot)$ and $\pi_\Omega(\cdot)$ are the operators of projection onto the positive orthant $\mathbb{R}^n_+$ and the set Ω, respectively, and $\nabla L_w(v, w, p) = \nabla \Phi_w(v, w) + \nabla g^\top(w) p$. Here $\nabla \Phi_w(v, w)$ is the gradient vector of the function $\Phi(v, w)$ with respect to the variable w and $\nabla g^\top(v)$ is the transposed matrix in which every column is the gradient vector of the corresponding scalar function of the vector $g(v)$.

We now consider a computation procedure in which two operations of projection onto a simple set Ω are implemented at each iteration as an auxiliary problem. This method is a forecast-type gradient technique [7]. Let v^0, p^0 be the initial approximation. Then the succeeding approximations can be computed by the recurrent formulas

$$\begin{aligned} \bar{p}^n &= \pi_+(p^n + \alpha g(v^n)), \\ \bar{u}^n &= \pi_\Omega(v^n - \alpha \nabla L_w(v^n, v^n, \bar{p}^n)), \\ v^{n+1} &= \pi_\Omega(v^n - \alpha \nabla L_w(\bar{u}^n, \bar{u}^n, \bar{p}^n)), \\ p^{n+1} &= \pi_+(p^n + \alpha g(\bar{u}^n)), \end{aligned} \tag{5.5}$$

where

$$L(v, w, p) = \Phi(v, w) + \langle p, g(w) \rangle, \qquad \nabla L_w(v, w, p) = \nabla \Phi_w(v, w) + \nabla g^\top(w) p.$$

For the operation of projection onto the set $D = \{w \mid g(w) \leq 0, \ \ w \in \Omega\}$, process (5.5) is of the form [7]

$$\begin{aligned} \bar{u}^n &= \pi_D(v^n - \alpha \nabla \Phi_w(v^n, v^n)), \\ v^{n+1} &= \pi_D(v^n - \alpha \nabla \Phi_w(\bar{u}^n, \bar{u}^n)). \end{aligned} \tag{5.6}$$

Below we shall show that this process converges under less stringent conditions than (5.5). Namely, for process (5.6) to converge, it is necessary that condition (3.3) be satisfied, whereas for process (5.5) to converge, it is necessary that the more stringent condition (3.7) or (3.8) be satisfied.

Let us express process (5.5) as variational inequalities. The first and fourth equations in (5.5), by the definition of the projection operator, can be expressed as

$$\langle \bar{p}^n - p^n - \alpha g(v^n), p - \bar{p}^n \rangle \geq 0 \qquad \forall p \geq 0, \tag{5.7}$$

and

$$\langle p^{n+1} - p^n - \alpha g(\bar{u}^n), p - p^{n+1}\rangle \geq 0 \qquad \forall p \geq 0, \tag{5.8}$$

respectively. Let us express the second and third equations as

$$\langle \bar{u}^n - v^n + \alpha \nabla L_w(v^n, v^n, \bar{p}^n), w - \bar{u}^n\rangle \geq 0 \qquad \forall w \in \Omega \tag{5.9}$$

and

$$\langle v^{n+1} - v^n + \alpha \nabla L_w(\bar{u}^n, \bar{u}^n, \bar{p}^n), w - v^{n+1}\rangle \geq 0 \qquad \forall w \in \Omega, \tag{5.10}$$

respectively.

In the sequel, we assume that the functions $\nabla\Phi_w(w, w)$, $\nabla g^\top(w)$, and $g(w)$ satisfy the Lipschitz conditions

$$|\nabla\Phi_w(w + h, w + h) - \nabla\Phi_w(w, w)| \leq |\nabla\Phi|\,|h|, \tag{5.11}$$
$$|\nabla g^\top(w + h) - \nabla g^\top(w)| \leq |\nabla g^\top|\,|h|, \tag{5.12}$$
$$|g(w + h) - g(w)| \leq |g|\,|h| \tag{5.13}$$

for all w and $w + h \in \Omega$, where $|\nabla\Phi|$, $|\nabla g^\top|$ and $|g|$ are Lipschitz constants.

Let us estimate the deviation of the vector v^{n+1} from the vector $\bar{u}^n$ and vice versa. Using the properties of the projection operator, estimates (5.11) and (5.12), and the boundedness condition $|p^n| \leq C$, from the second and third equations of (5.5), we obtain

$$\begin{aligned} & |v^{n+1} - \bar{u}^n| \leq \alpha|\nabla\Phi_w(\bar{u}^n, \bar{u}^n) + \nabla g^\top(\bar{u}^n)\bar{p}^n - \nabla\Phi_w(v^n, v^n) - \nabla g^\top(v^n)\bar{p}^n| \leq \\ \leq\ & \alpha|\nabla\Phi_w(\bar{u}^n, \bar{u}^n) - \nabla\Phi_w(v^n, v^n)| + \alpha|\nabla g^\top(\bar{u}^n)\bar{p}^n - \nabla g^\top(v^n)\bar{p}^n| \leq \\ \leq\ & \alpha|\nabla\Phi|\,|\bar{u}^n - v^n| + \alpha|\nabla g^\top|\,|\bar{p}^n|\,|\bar{u}^n - v^n| \leq \alpha(|\nabla\Phi| + |\nabla g^\top|C)|\bar{u}^n - v^n|. \end{aligned}$$

Hence,

$$|\bar{u}^n - v^{n+1}| \leq \alpha(|\nabla\Phi| + |\nabla g^\top|C)|\bar{u}^n - v^n|. \tag{5.14}$$

Let us also estimate the deviations of the vectors $\bar{p}^n$ and p^{n+1} from one another. From (5.5) and (5.13), we obtain

$$|\bar{p}^n - p^{n+1}| \leq \alpha|g|\,|\bar{u}^n - v^n|. \tag{5.15}$$

We now show that process (5.5) converges steadily in norm to one of the equilibrium solutions under the assumption that inequality (3.8) holds on Ω, i.e., on a set far wider than D. Since the sequence v^n belongs to Ω, this condition is trivial.

Theorem 1. *If the set of solutions of problem* (2.1) *is not empty and satisfies condition* (3.7) *or* (3.8)*, which hold on the set* Ω*, the aim function* $\Phi(v, w)$ *is continuous in* v *and convex in* w *for any* $v \in \Omega$*,* $\Omega \in \mathbb{R}^n$ *is a closed convex set, the functions* $\Phi(v, w)$ *and* $g(w)$ *satisfy conditions* (5.11) – (5.13)*, and* $|p^n| \leq C$*, then the sequence* v^n *generated by method* (5.5) *with parameter* $0 < \alpha < \dfrac{1}{\sqrt{2((|\nabla\Phi| + |\nabla g^\top|C)^2 + |g|^2)}}$ *converges steadily in norm to one of the equilibrium solutions, i.e.,* $v^n \to v^* \in \Omega^*$ *as* $n \to \infty$.

Proof. Take $w = v^*$ in (5.10). Then

$$\langle v^{n+1} - v^n + \alpha\nabla L_w(\bar{u}^n, \bar{u}^n, \bar{p}^n), v^* - v^{n+1}\rangle \geq 0. \tag{5.16}$$

Hence,

$$\begin{array}{rl} & \langle v^{n+1} - v^n, v^* - v^{n+1} \rangle + \alpha \langle \nabla L_w(\bar{u}^n, \bar{u}^n, \bar{p}^n), v^* - \bar{u}^n \rangle - \\ - & \alpha \langle \nabla L_w(v^n, v^n, \bar{p}^n) - \nabla L_w(\bar{u}^n, \bar{u}^n, \bar{p}^n), \bar{u}^n - v^{n+1} \rangle + \\ + & \alpha \langle \nabla L_w(v^n, v^n, \bar{p}^n), \bar{u}^n - v^{n+1} \rangle \geq 0. \end{array} \tag{5.17}$$

Since inequalities (3.6) are convex, we transform certain terms in (5.17). Let us first estimate the second term

$$\begin{array}{rl} & \langle \nabla L_w(\bar{u}^n, \bar{u}^n, \bar{p}^n), v^* - \bar{u}^n \rangle = \langle \nabla \Phi_w(\bar{u}^n, \bar{u}^n) + \nabla g^\top(\bar{u}^n)\bar{p}^n, v^* - \bar{u}^n \rangle \leq \\ \leq & \Phi(\bar{u}^n, v^*) - \Phi(\bar{u}^n, \bar{u}^n) + \langle \bar{p}^n, g(v^*) - g(\bar{u}^n) \rangle, \end{array}$$

and then the third term

$$\begin{array}{rl} & \langle \nabla L_w(v^n, v^n, \bar{p}^n) - \nabla L_w(\bar{u}^n, \bar{u}^n, \bar{p}^n), \bar{u}^n - v^{n+1} \rangle = \\ = & \langle \nabla \Phi_w(v^n, v^n) + \nabla g^\top(v^n)\bar{p}^n - \nabla \Phi_w(\bar{u}^n, \bar{u}^n) - \nabla g^\top(\bar{u}^n)\bar{p}^n, \bar{u}^n - v^{n+1} \rangle = \\ = & \langle \nabla \Phi_w(v^n, v^n) - \nabla \Phi_w(\bar{u}^n, \bar{u}^n), \bar{u}^n - v^{n+1} \rangle + \langle (\nabla g^\top(v^n) - \nabla g^\top(\bar{u}^n))\bar{p}^n, \bar{u}^n - v^{n+1} \rangle = \\ = & |\nabla \Phi| \, |v^n - \bar{u}^n| \, |\bar{u}^n - v^{n+1}| + |\nabla g^\top| v^n - \bar{u}^n| C |\bar{u}^n - v^{n+1}| \leq \\ \leq & \alpha(|\nabla \Phi| + |\nabla g^\top| C)^2 |\bar{u}^n - v^n|^2. \end{array}$$

Using these transformations, we represent (5.17) as

$$\begin{array}{rl} & \langle v^{n+1} - v^n, v^* - v^{n+1} \rangle + \alpha(\Phi(\bar{u}^n, v^*) - \Phi(\bar{u}^n, \bar{u}^n)) + \alpha \langle \bar{p}^n, g(v^*) - g(\bar{u}^n) \rangle + \\ + & \alpha^2(|\nabla \Phi| + |\nabla g^\top| C)^2 |\bar{u}^n - v^n|^2 + \alpha \langle \nabla L_w(v^n, v^n, \bar{p}^n), \bar{u}^n - v^{n+1} \rangle \geq 0. \end{array} \tag{5.18}$$

In inequality (5.9), let us take $w = v^{n+1}$. Then

$$\langle \bar{u}^n - v^n + \alpha \nabla L_w(v^n, v^n, \bar{p}^n), v^{n+1} - \bar{u}^n \rangle \geq 0. \tag{5.19}$$

Hence

$$\langle \bar{u}^n - v^n, v^{n+1} - \bar{u}^n \rangle + \alpha \langle \nabla L_w(v^n, v^n, \bar{p}^n), v^{n+1} - \bar{u}^n \rangle \geq 0. \tag{5.20}$$

Adding inequalities (5.18) and (5.20), we obtain

$$\begin{array}{rl} & \langle v^{n+1} - v^n, v^* - v^{n+1} \rangle + \alpha(\Phi(\bar{u}^n, v^*) - \Phi(\bar{u}^n, \bar{u}^n)) + \alpha \langle \bar{p}^n, g(v^*) - g(\bar{u}^n) \rangle + \\ + & \alpha^2(|\nabla \Phi| + |\nabla g^\top| C)^2 |\bar{u}^n - v^n|^2 + \langle \bar{u}^n - v^n, v^{n+1} - \bar{u}^n \rangle \geq 0. \end{array} \tag{5.21}$$

Taking $w = \bar{u}^n$, let us rewrite inequality (5.3) as

$$\Phi(v^*, v^*) + \langle p^*, g(v^*) \rangle \leq \Phi(v^*, \bar{u}^n) + \langle p^*, g(\bar{u}^n) \rangle. \tag{5.22}$$

Adding (5.21) and (5.22), by virtue of (3.8), we obtain

$$\begin{array}{rl} & \langle v^{n+1} - v^n, v^* - v^{n+1} \rangle + \langle \bar{u}^n - v^n, v^{n+1} - \bar{u}^n \rangle + \\ + & \alpha \langle \bar{p}^n - p^*, g(v^*) - g(\bar{u}^n) \rangle + \alpha^2(|\nabla \Phi| + |\nabla g^\top| C)^2 |\bar{u}^n - v^n|^2 \geq 0. \end{array} \tag{5.23}$$

Now let us repeat the above steps with inequalities (5.7) and (5.8). Take $p = p^*$ in (5.8)

$$\langle p^{n+1} - p^n, p^* - p^{n+1} \rangle - \alpha \langle g(\bar{u}^n), p^* - p^{n+1} \rangle \geq 0 \tag{5.24}$$

and take $p = p^{n+1}$ in (5.7)

$$\begin{aligned} &\langle \bar{p}^n - p^n, p^{n+1} - \bar{p}^n \rangle + \alpha \langle g(\bar{u}^n) - g(v^n), p^{n+1} - \bar{p}^n \rangle - \\ - \quad &\alpha \langle g(\bar{u}^n), p^{n+1} - \bar{p}^n \rangle \geq 0. \end{aligned} \tag{5.25}$$

Estimating the second term in the above inequality by (5.13) and (5.15) and then adding up inequalities (5.24) and (5.25), we obtain

$$\begin{aligned} &\langle p^{n+1} - p^n, p^* - p^{n+1} \rangle + \langle \bar{p}^n - p^n, p^{n+1} - \bar{p}^n \rangle + \\ + \quad &\alpha^2 |g|^2 \, |\bar{u}^n - v^n|^2 - \alpha \langle g(\bar{u}^n), p^* - \bar{p}^n \rangle \geq 0. \end{aligned} \tag{5.26}$$

Now, adding up (5.23) and (5.26), since $\langle \bar{p}^n, g(v^*) \rangle \leq 0$, $\langle p^*, g(v^*) \rangle = 0$, we obtain

$$\begin{aligned} &\langle v^{n+1} - v^n, v^* - v^{n+1} \rangle + \langle \bar{u}^n - v^n, v^{n+1} - \bar{u}^n \rangle + \langle p^{n+1} - p^n, p^* - p^{n+1} \rangle + \\ + \quad &\langle \bar{p}^n - p^n, p^{n+1} - \bar{p}^n \rangle + \alpha^2 \{ (|\nabla \Phi| + |\nabla g^\top| C)^2 + |g|^2 \} |\bar{u}^n - v^n|^2 \geq 0. \end{aligned} \tag{5.27}$$

Using the identity

$$|x_1 - x_3|^2 = |x_1 - x_2|^2 + 2\langle x_1 - x_2, x_2 - x_3 \rangle + |x_2 - x_3|^2, \tag{5.28}$$

let us expand the scalar products on the left side of the inequality as the sum of squares:

$$\begin{aligned} &|v^{n+1} - v^*|^2 + |p^{n+1} - p^*|^2 + |v^{n+1} - \bar{u}^n|^2 + d|\bar{u}^n - v^n|^2 + \\ + \quad &|p^{n+1} - \bar{p}^n|^2 + |\bar{p}^n - p^n|^2 \leq |v^n - v^*|^2 + |p^n - p^*|^2. \end{aligned} \tag{5.29}$$

By the conditions of the theorem, we have $d = 1 - 2\alpha^2 \{ (|\nabla \Phi| + |\nabla g^\top| C)^2 + |g|^2 \} > 0$. Summing inequality (5.29) from $n = 0$ to $n = N$, we obtain

$$\begin{aligned} &|v^{N+1} - v^*|^2 + |p^{N+1} - p^*|^2 + \sum_{k=0}^{k=N} |v^{k+1} - \bar{u}^k|^2 + d \sum_{k=0}^{k=N} |\bar{u}^k - v^k|^2 + \\ + \quad &\sum_{k=0}^{k=N} |p^{k+1} - \bar{p}^k|^2 + \sum_{k=0}^{k=N} |\bar{p}^k - p^k|^2 \leq |v^0 - v^*|^2 + |p^0 - p^*|^2. \end{aligned}$$

This inequality implies that the trajectory

$$|v^{N+1} - v^*|^2 + |p^{N+1} - p^*|^2 \leq |v^0 - v^*|^2 + |p^0 - p^*|^2$$

is bounded and the sequences

$$\sum_{k=0}^{\infty} |v^{k+1} - \bar{u}^k|^2 < \infty, \quad \sum_{k=0}^{\infty} |\bar{u}^k - v^k|^2 < \infty, \quad \sum_{k=0}^{\infty} |p^{k+1} - \bar{p}^k|^2 < \infty, \quad \sum_{k=0}^{\infty} |\bar{p}^k - p^k|^2 < \infty,$$

converge and, consequently,

$$|v^{n+1} - \bar{u}^n|^2 \to 0, \quad |\bar{u}^n - v^n|^2 \to 0, \quad |p^{n+1} - \bar{p}^n|^2 \to 0, \quad |\bar{p}^n - p^n|^2 \to 0, \qquad n \to \infty.$$

Since the sequence v^n, p^n is bounded, there exists an element v', p' such that $v^{n_i} \to v'$, $p^{n_i} \to p'$ as $n_i \to \infty$, and

$$|v^{n_i+1} - \bar{u}^{n_i}|^2 \to 0, \quad |\bar{u}^{n_i} - v^{n_i}|^2 \to 0, \quad |p^{n_i+1} - \bar{p}^{n_i}|^2 \to 0, \quad |\bar{p}^{n_i} - p^{n_i}|^2 \to 0.$$

Taking the limit in (5.5), for all $n_i \to \infty$, we obtain

$$v' = \pi_\Omega(v' - \alpha \nabla L_w(v', v', p')), \qquad p' = \pi_+(p' + \alpha g(v')).$$

Since these relations coincide with (5.4), we find that $v' = v^* \in \Omega^*$, $p' = p^* \geq 0$, i.e., every limit point of the sequence v^n, p^n is a solution of the problem. The monotonic decrement of $|v^n - v^*| + |p^n - p^*|$ guarantees the uniqueness of the limit, i.e., the convergence of $v^n \to v^*$, $p^n \to p^*$ as $n \to \infty$. $\square$

6. ECONOMIC INTERPRETATION

We examine the economic interpretation of the equilibrium problem and then the interpretation of the forecast method of solving this problem. For the sake of convenience in reasoning in terms of profit and cost, we assume that the objective function and the functional constraints in (2.1) are concave, i.e.,

$$v^* \in \text{Argmax}\{\Phi(v^*, w) \mid g(v^*, w) \geq b, \ \ w \in \Omega\}, \tag{6.1}$$

where $b \geq 0$. This problem represents the model of the interaction of two participants, one of which is called the controller, or simply control, and the other the production. Let $v \in \Omega$ be a set of alternatives of the first participant and let $w \in \Omega$ be a set of alternatives of the second participant. The move of the first participant consists in choosing $v \in \Omega$, which can be regarded as a production plan, production target, or production control. The response of production to a move of the first participant is always unique and consists in generating a set of solutions of problem (6.1). This set (maybe a singleton) consists of the optimal production plans for which the objective function $\Phi(v, w)$ for fixed v maximizes the income or profit from the products of this plan. Realization of one of the optimal plans is profitable for production and, accordingly, realization of the plan proposed by the control is not profitable. To resolve this conflict, the first participant must choose a control consistent with the optimal production plan. In formal terms, this is the fixed point of problem (6.1).

In this problem, the production objective function $\Phi(v, w)$ depends on the parameter $v \in \Omega$ which is controlled by the planning organ. If $v = v^0 \in \Omega$ is fixed, different profits can be obtained by varying the production plan $w \in \Omega$. In this situation, the normal response of production is to maximize its income. On the contrary, if the production level $w = w^0 \in \Omega$ is fixed, the planning organ changes the profit level by changing the control parameter $v \in \Omega$. The possibility at the disposal of the planning organ of regulating the production profit is defined to be the tax or surcharges paid to the planning organ by the production. In this situation, every value of the function $\Phi(v, w)$ on the diagonal of the square $\Omega \times \Omega$ is regarded as the compromise cost, because all equilibrium solutions lie on this diagonal.

In these representations, inequalities (3.1) and (3.3) have the following economic meaning. In terms of problem (6.1), inequality (3.1)

$$\Phi(v^*, v^*) \geq \Phi(v^*, w) \qquad \forall w \in \Omega$$

implies that in the equilibrium state it is not profitable for production to deviate from its optimal plan. Accordingly, inequality (3.3)

$$\Phi(w, v^*) \geq \Phi(w, w) \qquad \forall w \in \Omega$$

implies that in the equilibrium state, production tax must not be very high, i.e., the profit remaining after the payment of taxes must not be less than the compromise cost. If this condition is satisfied, the economic system will be stable (asymptotically stable).

This two-participant interaction scheme does not take account of the interest of the third participant, call the "market". Formally, the third participant is introduced into the game through the Lagrange function

$$L(v, w, p) = \Phi(v, w) + \langle p, g(w) - b \rangle \qquad \forall v, w \in \Omega \times \Omega, \ \forall p \geq 0. \tag{6.2}$$

The market acts on the economic system through the price $p \geq 0$. Production procures m types of resources $g(w)$ in these prices with regard for the changing stocks described by a vector b. If $g_i(w) > b_i$ for some $i = 1, 2, \ldots, m$, the superfluous resources remain after the realization of the plan w; but if $g_i(w) < b_i$, $i = 1, 2, \ldots, m$, then the initial resources are obviously insufficient and the lacking resources must be procured to complete the plan w. Every terms of the type $p_i(g_i(w) - b_i)$ in the Lagrange function, depending on the sign, denotes additional profit due to the same of superfluous resources or the expenses spent on procuring the lacking resources. The Lagrange function represented the aggregate profit resulting from the income due to the sale of the final product $\Phi(v, w)$ and either the addition profit due to the sale of superfluous resources or the expenses spent on the procurement of lacking resources. If the market dictates a certain price level, then production in response to the market policy, as in the case of a given control action, tends to maximize the aggregate profit, i.e., the Lagrange function [12].

The market derives its profit from the sale of resources to production under the condition that the latter have chosen its optimal plan $w \in \Omega$ for a given control $v \in \Omega$. In order to obtain maximum profit, the market minimize the Lagrange price function $\min\{L(v, w, p) \mid p \geq 0\}$ by varying the price $p \geq 0$. If there exists a saddle point w^*, p^* for some $v \in \Omega$, then we have the competing equilibrium

$$\min_{p \geq 0} L(v, w^* p) = \max_{w \in \Omega} L(v, w, p^*).$$

This relation equalizes the interests of production and market. In order to attain complete balance of interests on all participants in an economic system, the control and the production plan must be consistent with regard for the market trends.

Our method (5.5) is a mechanism for matching partially conflicting interests of the participants of an economic system. This method can be regarded as a multimove game with repeated gradient-type moves. Since production and market make the same type of moves, they can be easily predicted by the planning organ. Let the state of the economic system at the nth move be known and described by the vector of production output and price v^n, p^n. A the succeeding move, the plan makes a price forecast and computes the vector $\bar{p}^n$ by formulas (5.5) (first row). Then it computes the production output forecast $\bar{u}^n$ with reference to the price forecast. This forecast is passed on to production as a target. Production then realizes this target and computes the real vector of production output by the formula

$$v^{n+1} = \pi_\Omega(v^n - \alpha \nabla L_w(\bar{u}^n, v^n, \bar{p}^n)).$$

The effectiveness of this move will be slightly higher if production is based not on the real output vector v^n, but on the forecast $\bar{u}^n$. Precisely, this variant is at the base of formulas (5.5) (third row). At the final $(n+1)$th iteration, the market computes the real price vector p^{n+1} on the basis of real v^{n+1} or forecast $\bar{u}^n$ production output. The control behavior strategy based on forecast is effective, because the game converges to the equilibrium state of the economic system with increasing number of moves.

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